BOOK REVIEWS

Generalized functions and partial differential equations. By Avner Friedman. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1963. xii+340 pp. $10.00.

This book deals with applications of Fourier transformation to certain problems of linear partial differential equations. Fourier transformation is meant here in the sense of the theory of distributions and of generalized functions. The author follows rather closely the original presentations of these theories by Laurent Schwartz and Gel′fand and Shilov. Chapter 1 provides the functional analysis background, giving the elements of the theory of topological vector spaces along the lines of N. Bourbaki, *Espaces vectoriels topologiques*, Paris, 1953–1955. Use is made of a subclass of $\mathfrak{F}$-spaces, under the name of complete countably normed. Perfect spaces are complete countably normed spaces in which every closed bounded set is compact. The reviewer has not found a single nontrivial property of the complete countably normed spaces which does not extend to $\mathfrak{F}$-spaces and it seems to him that this new definition complicates needlessly the picture. Chapter 2 introduces the two main classes of spaces of test functions. The spaces of the first kind, denoted by $K\{M_m\}$, consist of C^∞ functions ϕ on R^n satisfying inequalities of the kind

$$\sup_{|p|\leq m} \sup_{x} M_m(x) \mid D^p\phi(x) \mid \ < +\infty.$$

The weight functions $M_m(x)$ are allowed to be infinite on certain sets, in which $D^p\phi$ must then vanish; the sup with respect to x is performed over the complement of these sets. The elements of the spaces of the second kind, called $Z\{M_m\}$, are essentially entire functions on C^n, satisfying a sequence of inequalities

$$\sup_{z\in C^n} M_m(z) \mid \phi(z) \mid \ < +\infty.$$

Here the M_m form a nondecreasing sequence of positive continuous functions in C^n. Further spaces of test functions are obtained by taking the intersections and the inductive limits of the spaces above.

Chapter 2 proceeds then with the definition of generalized functions and the study of their basic properties. A generalized function is a continuous linear form over a space of test functions. A first example is provided by the distributions; the main spaces of test functions for distributions, the spaces $\mathfrak{D}$, $\mathcal{E}$, $\mathcal{S}$ are of the type $K\{M_m\}$ (or inductive limits of spaces $K\{M_m\}$). Chapter 3 constitutes a summarized version of the book of L. Schwartz, *Théorie des distributions*, Paris, 1950–1951. Chapter 4 extends certain properties of the convolution and the Fourier transform of distributions to larger classes of spaces of generalized functions. Chapter 5 is devoted to the study of the so-called W spaces. These are intersections (as the parameters a_i' below vary in open intervals $]0, a_i[$) of spaces $K\{M_m\}$ in which all the functions M_m are equal to the same function

$$\prod_{i=1}^{n} \exp g_i(a_i' \mid x_i \mid).$$

The exponents g_i are suitable convex functions. There is a second class of W spaces, the obvious analogue for entire functions on C^n. To establish the needed properties of these W spaces one makes use of various theorems about entire functions which are proved in the same chapter. The following chapter is also devoted to results about entire functions (always of finite order), related with properties of their Fourier transforms (which are generalized functions). Theorems of the Paley-Wiener type are proved. With the W spaces the author has in his hands the tools which enable him to solve a wide class of Cauchy problems in a half space. Later on, further applications of the theory of generalized functions require the introduction of additional functional spaces, in Chapter 9, under the name of S spaces.

Chapter 7 is the core of the book, justifying the introduction of the huge machinery of the previous chapters. It is concerned with solving the problem

$$(1) \qquad \left[\frac{\partial}{\partial t} - P\left(t, \frac{\partial}{\partial x}\right)\right] u(x, t) = f(x, t), \qquad x \in R^n, 0 < t \leqq T,$$

$$(2) \qquad u(x, 0) = u_0(x), \qquad x \in R^n.$$

Here u, f, u_0 are vectors with generalized functions as components; $P(t, \partial/\partial x)$ is a matrix whose entries are linear partial differential operators in x with coefficients independent of x. The latter property allows to perform a Fourier transformation with respect to x and therefore to deal with an ordinary differential system in t having coefficients which depend polynomially on "parameters" σ. The first part of the chapter studies the case of constant coefficients, that is to say, independent also of t. A number of uniqueness theorems are proved when data and solutions belong, for each t, to duals of suitable W spaces (in the variable x). These W spaces are defined by regularity and growth conditions. In relation with the existence theorems are introduced the types: parabolic, hyperbolic, correctly posed systems.

The various results on existence and uniqueness of solutions to the Cauchy problems (1), (2) are then extended to the case of coefficients which are continuous functions of t. Moreover, under suitable hypotheses of regularity and growth at infinity on the data (and of type on the operators), the uniqueness and existence of classical solutions is established. In the last sections of Chapter 7 the author gives necessary conditions on the differential operators for the Cauchy problem to be correctly posed. These are results of the kind first proved by L. Gårding with the help of the theorem of Seidenberg-Tarski (of which a proof is given). Extensions of the methods developed in Chapter 7 to problems involving several time variables make up the content of Chapter 8.

After Chapter 9, devoted as we have said to the S spaces, two more chapters contain applications of the Fourier transformation of distributions to partial differential equations with constant coefficients. Among other results the existence of fundamental solutions is proved, the hypoelliptic differential polynomials are characterized, the existence of solutions of the inhomogeneous equation in the whole space R^n is studied. On these topics the student has now at his disposal the more systematic and comprehensive exposition of L. Hörmander, *Linear partial differential operators*, Springer-Verlag, Berlin, 1963.

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