NONLINEAR PARABOLIC BOUNDARY VALUE PROBLEMS OF ARBITRARY ORDER[1]

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In a number of recent papers the writer has developed a new nonlinear version of the orthogonal projection argument to prove the existence of solutions of variational type for boundary value problems for nonlinear partial differential equations, first for nonlinear elliptic equations in [2], [3], [4], [5], and [6] and more recently for nonlinear parabolic equations in [7]. This new method is based on general theorems on the existence of solutions of equations in Hilbert space or reflexive Banach spaces involving operators satisfying very weak continuity conditions but having suitable monotonicity properties.[2] Our results in [6] in the elliptic case included much stronger versions of theorems of Visik ([12], [13], [14]) on solutions of boundary value problems for equations of the form

$$(1) \qquad Au = \sum_{|\alpha| \leq m} D^\alpha A_\alpha(x, u, \cdots, D^m u) = f(x),$$

with the A_α having polynomial growth in $(u, \cdots, D^m u)$.[3] In the parabolic case, we considered in [7] mixed initial-boundary value problems for equations of the form

$$(2) \qquad \frac{\partial u}{\partial t} + \sum_{|\alpha| \leq m} D^\alpha A_\alpha(x, t, u, \cdots, D^m u) = f(x, t)$$

with A_α having at most linear growth in $(u, \cdots, D^m u)$.

In the present note, we present in summary a complete generalization of the treatment of the parabolic problems to the case of A_α of polynomial growth in $(u, \cdots, D^m u)$. The detailed arguments will appear in [8]. This generalization is based upon a stronger theorem in reflexive Banach spaces, Theorem 2 below, which generalizes the abstract theorems given in both [6] and [7].

We use the notation and definitions of [6] and [7]. We let Ω be an

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[2] Earlier but weaker results on continuous monotone operator equations were obtained by Vainberg and Kachurovski [11] and G. J. Minty [10].

[3] For the case of linear growth and the Euler equations of a regular variational problem, an interesting generalization of the Morse Theory has been obtained recently by S. Smale and by Smale and R. Palais.

open set of R^n whose points are denoted by $x=(x_1, \cdots, x_n)$, R^1 having coordinate t, and (x, t) a point of $\Omega\times R^1$. D^α will denote differential operators with respect to $x=(x_1, \cdots, x_n)$ and $\partial/\partial t$ differentiation with respect to t. Let S be a segment $[t_0, t_1]\subset R^1$.

Let $A(t)$ be a system of differential operators of the form

$$A(t)u = \sum_{|\alpha|\leq m} D^\alpha A_\alpha(x, t, u, \cdots, D^m u) \tag{1}$$

where we assume the following:

Assumption I. *Each A_α is an r-vector function of (x, t) in $\Omega\times S$, of the value of the r-vector function u at (x, t) and of the values of $D^\beta u$ for $|\beta|\leqq m$.*

A_α is measurable in (x, t) and continuous in $(u, \cdots, D^m u)$.

For a fixed $p>1$, there exists a constant c and a function $g\in L^q(\Omega)$ with $q=p(p-1)^{-1}$ such that for all (x, t) in Ω and each complex vector $\zeta=\{\zeta_\beta; |\beta|\leqq m\}$

$$|A_\alpha(x, t, \zeta)| \leqq c \sum_{|\beta|\leq m} |\zeta_\beta|^{p-1} + g(x).$$

Let V be a closed subspace of $W^{m,p}(\Omega)$, where p is the real number of Assumption I, and $C_c^\infty(\Omega)\subset V$. Assumption I implies that for every u and v in $L^p(S; V)$, the nonlinear Dirichlet form given by

$$h(u, v) = \int_{t_0}^{t_1} \langle A_\alpha(x, u(t), \cdots, D^m u(t)), D^\alpha v(t)\rangle dt \tag{2}$$

is well defined. (We recall that $\langle w, u\rangle$ is the L^2-inner product on Ω.)

Let

$$F_0 = \left\{ v \,\middle|\, \begin{array}{l} v \text{ is continuous from } S \text{ to } V, \\ v \text{ is } C^1 \text{ from } S \text{ to } L^q(\Omega),\ q = p(p-1)^{-1},\ v(t_0) = 0. \end{array} \right\}$$

We know from Assumption I that for each u in $L^p(S, V)$

$$|h(u, v)| \leqq (\|u\|)\|v\|$$

where

$$\|u\| = \left\{ \int_{t_0}^{t_1} \|u\|_{m,p}^p dt \right\}^{1/p}.$$

If $l(u, v)$ is the sesquilinear form defined for u in F_0, $v\in L^2(S, V)$ by

$$l(u, v) = \int_{t_0}^{t_1} \langle \frac{du}{dt}(t), v(t)\rangle dt,$$

then

$$|l(u, v)| \leqq c(u)\|v\|.$$

Hence there exists a unique element $L_0 u$ in $(L^p(S, V))^*$ such that

$$(L_0 u, v) = l(u, v).$$

DEFINITION. *L is the closed densely-defined linear operator from $L^p(S, V)$ to $(L^p(S, V))^*$ which is the closure of L_0.*

VARIATIONAL PARABOLIC BOUNDARY VALUE PROBLEM. *Given $f \in (L^p(S, V))^*$, to find u in $L^p(S, V)$ with $u \in D(L)$ such that*

$$(Lu, v) + h(u, v) = (f, v)$$

for all v in $L^p(S, V)$.

THEOREM 1. *Suppose $A(t)$ satisfies Assumption* I *and both of the following conditions:*

(a) *For all $u, v \in L^p(S, V)$,*

$$\operatorname{Re}\{h(u, u - v) - h(v, u - v)\} \geqq 0.$$

(b) *There exists a non-negative function $c(r)$ with $c(r) \to +\infty$ as $r \to +\infty$ such that*

$$\operatorname{Re} h(u, u) \geqq c(\|u\|)\|u\|$$

for all $u \in L^p(S, V)$.

Then there exists one and only one solution u of the variational boundary value problem for the equation

$$\frac{\partial u}{\partial t} + A(t)u = f$$

for given f in $(L^p(S, V))^*$. This solution u is continuous from S to $L^2(\Omega)$ and $u(t_0) = 0$.

The proof of Theorem 1 rests upon the following general theorem for operators in a reflexive separable Banach space.

THEOREM 2. *Let X be a reflexive separable Banach space, X^* its conjugate space, (w, u) the pairing between an element w of X^* and u of X. Let T be an operator, not necessarily linear, with dense domain $D(T)$ in X and range $R(T)$ in X^*. Suppose that $T = L + G$ where*

(a) *L is a closed densely defined linear operator from X to X^* such that if $W = D(L) \cap D(L^*)$, then L is the closure of its restriction to W and L^* is the closure of its restriction to W.*

(b) *G is defined on all of X, maps bounded sets of X into bounded sets of X^*, and G is demi-continuous (i.e. continuous from the strong topology of X to the weak topology of X^*).*

Suppose further that:

(c) *There exists a compact mapping C from X to X^* such that for all u and v of $D(T)$*

$$\operatorname{Re}(Tu - Tv, u - v) \geqq -\operatorname{Re}(Cu - Cv, u - v).$$

(d) *There exists a continuous real-valued function $c(r)$ with $c(r) \to +\infty$ as $r \to +\infty$ such that for all u in $D(T)$*

$$\operatorname{Re}(Tu, u) \geqq c(\|u\|)\|u\|.$$

Then T is onto, i.e. for each w in X^* there exists u in X such that $Tu = w$.

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