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# DOUBLY INVARIANT SUBSPACES OF ANNULUS OPERATORS

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**1. Introduction.** Let $C$ be the unit circle in the complex plane and let $C_0$ be the circle $\{z: |z| = r_0\}$, where $r_0$ is a positive real number less than unity. The set $C \cup C_0$ is the boundary of the annulus $A = \{z: r_0 < |z| < 1\}$. Let us endow the circles $C$ and $C_0$ with Lebesgue measure of total mass unity, and denote by $L^2(\partial A)$ the $L^2$ space associated with the measure thereby defined on the set $C \cup C_0$. This note concerns the invariant subspaces of the position operator on the space $L^2(\partial A)$, that is, of the operator $Z$ on $L^2(\partial A)$ defined by $(Zx)(z) = zx(z)$.

We may regard $L^2(\partial A)$ as the direct sum of the two spaces $L^2(C)$ and $L^2(C_0)$. As subspaces of $L^2(\partial A)$, the latter reduce the operator $Z$. The restriction of $Z$ to $L^2(C)$ is a well-known operator, a so-called bilateral shift (of unit multiplicity). The invariant subspaces of this operator have been extensively studied by Beurling [**1**], by Helson and Lowdenslager [**3**], and by Halmos [**2**]. The restriction of $Z$ to $L^2(C_0)$ is a bilateral shift multiplied by the scalar $r_0$, and so has the same invariant subspace structure as a bilateral shift. The operator $Z$ is therefore the direct sum of two operators whose invariant subspaces have been completely described. However, the problem of determining the invariant subspaces of $Z$ involves more than merely a routine extension of known results about bilateral shifts, and as yet has not been solved completely.

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The purpose of this note is to announce results concerning the invariant subspaces of $Z$ that are also invariant under $Z^{-1}$; these we call *doubly invariant subspaces* of $Z$. The proofs, to appear elsewhere, depend to a large extent on complex function theory, and involve analogues for functions in the $\mathbf{H}^2$ space of the annulus $A$ of a number of well-known properties of functions in the $\mathbf{H}^2$ space of a disk.

2. **A characterization of doubly invariant subspaces.** For every real number $\alpha$ we define the function $w_\alpha$ on $C\cup C_0$ by

$$\left.\begin{aligned} w_\alpha(e^{it}) &= e^{i\alpha t} \\ w_\alpha(r_0e^{it}) &= r_0^\alpha e^{i\alpha t} \end{aligned}\right\} \quad 0 \leqq t < 2\pi.$$

Then $Zw_\alpha = w_{\alpha+1}$, and the functions $w_\alpha$ and $w_{\alpha+n}$ are orthogonal for $n = \pm 1, \pm 2, \cdots$. For $0 \leqq \alpha < 1$ we denote by $H^2_\alpha(\partial A)$ the smallest doubly invariant subspace of $Z$ containing $w_\alpha$, that is, the span in $\mathbf{L}^2(\partial A)$ of the functions $w_{\alpha+n}$, $n = 0, \pm 1, \pm 2, \cdots$. More generally, for any function $x$ in $\mathbf{L}^2(\partial A)$ we let $\mathbf{M}_x$ denote the smallest doubly invariant subspace of $Z$ containing $x$. The subspaces $\mathbf{H}^2_\alpha(\partial A)$ are prototypes of the doubly invariant subspaces that do not reduce $Z$. More precisely, we have the following two theorems.

THEOREM 1. *Let $x$ be a function in $\mathbf{L}^2(\partial A)$. If the condition*

$$(*) \qquad \int_0^{2\pi} \log |x(e^{it})| \, dt + \int_0^{2\pi} \log |x(r_0e^{it})| \, dt > -\infty$$

*is satisfied, let $\alpha$ be the number in the interval* $[0, 1)$ *congruent modulo* 1 *to the number*

$$\frac{1}{2\pi q_0}\left[\int_0^{2\pi} \log |x(e^{it})| \, dt - \int_0^{2\pi} \log |x(r_0e^{it})| \, dt\right],$$

*where $q_0 = -\log r_0$. Then there is a measurable function $w$ on $C\cup C_0$, with $|w| = 1$ almost everywhere, such that $\mathbf{M}_x$ consists of all products $wy$ with $y$ in $\mathbf{H}^2_\alpha(\partial A)$. The function $w$ is unique to within a multiplicative constant of unit modulus.*

*On the other hand, if condition* (*) *is not satisfied (that is, if the function $x$ is "small"), then $\mathbf{M}_x$ consists of all functions in $\mathbf{L}^2(\partial A)$ that vanish at every point where $x$ vanishes.*

THEOREM 2. *If $\mathbf{M}$ is any doubly invariant subspace of $Z$, then there is a function $x$ such that $\mathbf{M} = \mathbf{M}_x$.*

Theorems 1 and 2 characterize the doubly invariant subspaces of $Z$. We see in particular that if $x$ is a function in $\mathbf{L}^2(\partial A)$ satisfying (*), and if $\alpha$ is as defined in Theorem 1, then the operator $Z|\boldsymbol{M}_x$ is unitarily equivalent to $Z|\boldsymbol{H}^2_\alpha(\partial A)$. On the other hand, one can show that the operators $Z|\boldsymbol{H}^2_\alpha(\partial A)$ and $Z|\boldsymbol{H}^2_\beta(\partial A)$ are not unitarily equivalent for $\alpha \neq \beta$.

3. **Doubly invariant subspaces in $\boldsymbol{H}^2(A)$.** The space $\boldsymbol{H}^2(A)$ consists by definition of all holomorphic functions $f$ in the annulus $A$ such that

$$\sup_{r_0<r<1} \int_0^{2\pi} |f(re^{it})|^2 dt < \infty.$$

Just as in a disk, a function of class $\boldsymbol{H}^2(A)$ has nontangential limits at almost every boundary point of $A$, and so can be extended (almost everywhere) to the boundary. By Fatou's lemma, the resulting boundary function belongs to $\boldsymbol{L}^2(\partial A)$. In fact, one can show that the boundary function belongs to $\boldsymbol{H}^2_0(\partial A)$. Conversely, any function in $\boldsymbol{H}^2_0(\partial A)$ is the boundary function of a unique function in $\boldsymbol{H}^2(A)$. The spaces $\boldsymbol{H}^2_0(\partial A)$ and $\boldsymbol{H}^2(A)$ are thus in one-to-one correspondence, and the latter is thereby endowed with a Hilbert space structure. The operator $Z|\boldsymbol{H}^2_0(\partial A)$ corresponds to the operator $Z_0$ on $\boldsymbol{H}^2(A)$ defined by $(Z_0 f)(z) = zf(z)$. We conclude with two results concerning doubly invariant subspaces of the operator $Z_0$.

THEOREM 3. *Let $a_1, a_2, a_3, \cdots$ be a finite or infinite sequence of points in the annulus $A$ (repetitions allowed). Let $\boldsymbol{M}$ be the collection of all functions in $\boldsymbol{H}^2(A)$ that vanish (with the appropriate multiplicity) at each point $a_k$. Then $\boldsymbol{M}$ is a doubly invariant subspace of $Z_0$. If the sum*

$$(**) \qquad \sum \min\left(1 - |a_k|,\ 1 - \frac{r_0}{|a_k|}\right)$$

*is infinite, then $\boldsymbol{M}$ is trivial. If the sum (**) is finite, then $Z_0|\boldsymbol{M}$ is unitarily equivalent to $Z|\boldsymbol{H}^2_\alpha(\partial A)$, where $\alpha$ is the number in the interval $[0, 1)$ congruent modulo 1 to the number $\sum \alpha_k$, the $\alpha_k$ being defined by*

$$\alpha_k = \begin{cases} \dfrac{-1}{q_0} \log |a_k| & \text{if } r_0^{1/2} \leqq |a_k| < 1, \\[2ex] \dfrac{-1}{q_0} \log (|a_k|/r_0) & \text{if } r_0 < |a_k| < r_0^{1/2}, \end{cases}$$

$q_0 = -\log r_0$.

We shall say that a function in $\boldsymbol{H}^2(A)$ is a cyclic vector of $Z_0$ if it is contained in no proper doubly invariant subspace of $Z_0$.

THEOREM 4. *A function $f \neq 0$ in $H^2(A)$ is a cyclic vector of $Z_0$ if and only if it satisfies the condition*

$$\int_0^{2\pi} \log |f(r_0^{\delta} e^{it})| \, dt = \delta \int_0^{2\pi} \log |f(r_0 e^{it})| \, dt + (1-\delta) \int_0^{2\pi} \log |f(e^{it})| \, dt \qquad \text{for } 0 < \delta < 1.$$

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