TWO THEOREMS CONCERNING FUNCTIONS HOLOMORPHIC ON MULTIPLY CONNECTED DOMAINS

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Communicated by Walter Rudin, March 25, 1963

1. Let Ω be a finitely connected plane domain whose boundary, $\partial\Omega$, consists of the circles $\Gamma_0, \Gamma_1, \cdots, \Gamma_n$. We assume Γ_j lies in the interior of Γ_0 for $j=1, 2, \cdots, n$. Let Δ_0 be the interior of Γ_0 and let Δ_j be the exterior of Γ_j, $j=1, 2, \cdots, n$. We then have $\Omega=\bigcap_{j=0}^{n} \Delta_j$. Let $H_\infty[\Omega]$ be the collection of all bounded holomorphic functions in Ω. We shall say that a set S of points of Ω is an interpolation set for Ω if given a bounded complex valued function w on S there is $f\in H_\infty[\Omega]$ such that $f(z)=w(z)$ for all $z\in S$. If $\{z_n\}_{n=1}^{\infty}$ is a sequence in Ω, without limit points in Ω, we write $\{z_n\}=S_0\cup S_1\cup \cdots \cup S_n$ where the S_j are pairwise disjoint and where the only limit points of S_j lie in Γ_j, $j=0, 1, \cdots, n$.

In the present note we sketch proofs for the following two theorems:

THEOREM A. *The sequence $\{z_n\}$ is an interpolation set for Ω if and only if each S_j is an interpolation set for the disc Δ_j.*

THEOREM B. *Let $f_1, f_2, \cdots, f_m$ be functions in $H_\infty[\Omega]$ such that $|f_1(z)|+|f_2(z)|+\cdots+|f_m(z)|\geqq\delta>0$ for all $z\in\Omega$. Then there exist functions $g_1, g_2, \cdots, g_m\in H_\infty[\Omega]$ such that $f_1g_1+f_2g_2+\cdots+f_mg_m=1$.*

L. Carleson [2] has established Theorem B in case Ω is the open unit disc. He has also proved [1] that the sequence $\{z_n\}_{n=1}^{\infty}$ is an interpolation sequence for the open unit disc if and only if there is a $\delta>0$ such that

$$\prod_{n\neq k}\left|\frac{z_n-z_k}{1-\bar{z}_nz_k}\right|>\delta$$

for $k=1, 2, 3, \cdots$. For a discussion and alternative proof see [**3**, pp. 194–208].

2. **Outline of the proof of Theorem A.** Let B_j be the Blaschke product associated with the disc Δ_j and the set of points S_j, $j=0, \cdots, n$. Note that there is an $\eta>0$ such that $|B_j(z)|>\eta$ for $z\in S_k$ if $k\neq j$.

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Suppose now that S_j is an interpolation set for Δ_j, $j=0, \cdots, n$. Let w be a bounded function on S, and let $f_j \in H_\infty[\Delta_j]$ be such that

$$f_j(z) = w(z)/(B_0(z) \cdots B_{j-1}(z)B_{j+1}(z) \cdots B_n(z))$$

for all $z\in S_j$. Define

$$F = f_0B_1B_2 \cdots B_n + f_1B_0B_2 \cdots B_n + \cdots + f_nB_0B_1 \cdots B_{n-1}.$$

Then $F\in H_\infty[\Omega]$ and $F(z)=w(z)$ for all $z\in S$.

Conversely, assume that $\{z_n\}_{n=1}^\infty$ is an interpolation set for Ω. If $f\in H_\infty[\Omega]$ we define $\|f\|$ by

$$\|f\| = \sup\{ |f(z)| : z \in \Omega\}. \tag{1}$$

A Banach space argument like that in [**3**, p. 196] shows that there is a constant M such that if w is a function on $\{z_n\}_{n=1}^\infty$ with $|w(z)| \leqq 1$ for all $z\in \{z_n\}_{n=1}^\infty$, then there is $f\in H_\infty[\Omega]$ with $\|f\| \leqq M$ and $f(z)=w(z)$, $z\in\{z_n\}_{n=1}^\infty$. Given $z_k\in S_j$, let $B_j^{(k)}$ be the Blaschke product associated with the disc Δ_j and the set $S_j\backslash\{z_k\}$. Let $f\in H_\infty[\Omega]$ be such that $f(z_n)=0$ for $n\neq k$, $f(z_k)=1$ and such that $\|f\|\leqq M$. The function

$$g = f/(B_0 \cdots B_{j-1}B_j^{(k)} B_{j+1} \cdots B_n)$$

is in $H_\infty[\Omega]$. Since there is $\delta>0$ such that $|B_i(z)|\geqq\delta$ for all $z\in\Gamma_j$, $i\neq j$, we have that $\|g\|\leqq M/\delta^n$. In particular then $|g(z_k)|\leqq M/\delta^n$. This yields

$$| \delta^n M^{-1} f(z_k)/(B_0(z_k) \cdots B_{j-1}(z_k)B_{j+1}(z_k) \cdots B_n(z_k)) | \leqq | B_j^{(k)}(z_k) | .$$

Since $f(z_k)=1$, and the product $B_0 \cdots B_{j-1}B_{j+1} \cdots B_n$ is uniformly bounded away from zero on S_j, we have that $B_j^{(k)}(z_k)\geqq\delta_1>0$. This estimate is uniform in k, so S_j is an interpolation set for Δ_j.

3. **Outline of the proof of Theorem B.** Observe that $H_\infty[\Omega]$ is a commutative Banach algebra with identity if it is given the norm defined by (1). Let $\mathfrak{M}[\Omega]$ be the maximal ideal space of $H_\infty[\Omega]$; we regard $\mathfrak{M}[\Omega]$ as the collection of all nonzero complex homomorphisms of $H_\infty[\Omega]$ with the weak* topology. Let $\mathfrak{M}_e[\Omega]$ be the collection of those homomorphisms ϕ_λ of the form $\phi_\lambda(f)=f(\lambda)$, $\lambda\in\Omega$. It is known [**3**, p. 163] that to establish our result it suffices to prove $\mathfrak{M}_e[\Omega]$ dense in $\mathfrak{M}[\Omega]$.

For $j=1, 2, \cdots, n$, let $H_\infty^0[\Delta_j]$ be the closed subalgebra of $H_\infty[\Omega]$ consisting of those f which are restrictions to Ω of functions in $H_\infty[\Delta_j]$ which vanish at infinity. It is known [**4**, p. 56] that if $f\in H_\infty[\Omega]$, then f can be written in the form

$$f = f_0 + f_1 + \cdots + f_n, \tag{2}$$

$f_0 \in H_\infty[\Delta_0]$, $f_j \in H_\infty^0[\Delta_j]$, $1 \leqq j$. It is immediate that this decomposition is unique; it yields

$$H_\infty[\Omega] = H_\infty[\Delta_0] \oplus H_\infty^0[\Delta_1] \oplus \cdots \oplus H_\infty^0[\Delta_n], \tag{3}$$

the direct sum being understood in the sense of Banach spaces.

Following some ideas of I. J. Schark (see [3, p. 159, ff.]), we note that the function z is in $H_\infty[\Omega]$. It gives rise to the function $\hat{z}$ on $\mathfrak{M}[\Omega]$ given by $\hat{z}(\phi) = \phi(z)$. We can prove that $\hat{z}$ maps $\mathfrak{M}[\Omega]$ onto $\bar{\Omega}$ and that $\hat{z}$ is one-to-one over Ω. If $\alpha \in \partial\Omega$, set $\mathfrak{M}_\alpha = \{\phi \in \mathfrak{M}[\Omega]: \hat{z}(\phi) = \alpha\}$. A slight modification of the argument for the disc case shows that if $f \in H_\infty[\Omega]$, then $\hat{f}$ is constant on $\mathfrak{M}_\alpha$ if and only if f is continuously extensible to $\Omega \cup \{\alpha\}$ and that if f is so extensible, then $\hat{f}(\phi) = f(\alpha)$ for all $\phi \in \mathfrak{M}[\Omega]$.

Suppose now that ϕ is a multiplicative linear functional defined on $H_\infty[\Delta_0]$ viewed as a subalgebra of $H_\infty[\Omega]$ by the direct sum decomposition (3). Let $\phi(z) \in \Omega$. Then ϕ admits a unique extension to an element of $\mathfrak{M}[\Omega]$. This is clear since $\hat{z}$ maps $\mathfrak{M}[\Omega]$ onto $\bar{\Omega}$ and is one-to-one over Ω. If $\alpha = \phi(z)$ lies in Γ_0, ϕ also admits a unique extension to an element of $\mathfrak{M}[\Omega]$. For uniqueness, suppose that ϕ^* is an extension of ϕ to all of $H_\infty[\Omega]$. For $f \in H_\infty[\Omega]$, write $f = f_0 + f_1 + \cdots + f_n$ in accordance with (2). The linearity of ϕ^* implies that $\phi^*(f) = \phi^*(f_0) + \phi^*(f_1) + \cdots + \phi^*(f_n)$. Since ϕ^* is an extension of ϕ, and since, for $j = 1, 2, \cdots, n$, f_j is continuously extensible to $\Omega \cup \{\alpha\}$, it follows that $\phi^*(f) = \phi(f_0) + f_1(\alpha) + \cdots + f_n(\alpha)$. This establishes the uniqueness of the extension. This choice of ϕ^* yields a multiplicative functional. To see this, suppose $g \in H_\infty[\Omega]$ and write $g = g_0 + g_1 + \cdots + g_n$ by (2). Then $fg = \sum_{j,k=0}^{n} f_j g_k$. Since ϕ^* is plainly linear, we need only show $\phi^*(f_j g_k) = \phi^*(f_j)\phi^*(g_k)$. If neither j nor k is zero, $f_j g_k$ is continuously extensible to $\Omega \cup \{\alpha\}$, so we need only consider terms of the form $f_0 g_k$ and $f_j g_0$. Suppose then that $f \in H_\infty[\Delta_0]$, $g \in H_\infty[\Delta_j]$, $j \neq 0$. Since $\phi^*(g) = g(\alpha)$, we are finished if we can show $\phi^*(fg - g(\alpha)f) = 0$. Write

$$fg - g(\alpha)f = h_0 + h_1 + \cdots + h_n$$

in accordance with (2). Then h_j is continuous at α for $j = 1, \cdots, n$, and since $fg - g(\alpha)f$ is continuous at α, it follows that h_0 must be continuous at α so that $\phi(h_0) = h_0(\alpha)$. Therefore $\phi^*(fg - g(\alpha)f) = h_0(\alpha) + h_1(\alpha) + \cdots + h_n(\alpha) = 0$. We conclude that ϕ^* is multiplicative.

If ϕ is a multiplicative linear functional on $H_\infty[\Delta_0]$ such that $\phi(z) \in \Gamma_j$ for $j \neq 0$, our argument indicates that ϕ admits many extensions to an element of $\mathfrak{M}[\Omega]$. If $\phi(z)\in\Delta_0\backslash\bar{\Omega}$, then ϕ admits no extension.

The same argument applies to $\Delta_1, \cdots, \Delta_n$ in place of Δ_0. This also shows that every element of $\mathfrak{M}[\Omega]$ is determined by its action on the subalgebras $H_\infty[\Delta_0]$, $H_\infty^0[\Delta_1], \cdots, H_\infty^0[\Delta_n]$. It now follows that $\mathfrak{M}_e[\Omega]$ is dense in $\mathfrak{M}[\Omega]$. For suppose $\phi\in\mathfrak{M}[\Omega]$, and suppose $\alpha=\phi(z)\in\Gamma_k$. Let $\phi^{(j)}$ be the restriction of ϕ to the subalgebra $H_\infty[\Delta_j]$. By Carleson's result for the disc, there is a point $\lambda\in\Delta_k$ such that the point evaluation $\phi_\lambda^{(k)}$ at λ is near $\phi^{(k)}$ in the sense of the weak* topology in the maximal ideal space of $H_\infty[\Delta_k]$. If λ is near α, then $\lambda\in\Omega$, and each of the point evaluations at λ, $\phi_\lambda^{(j)}$, for $j\neq k$ is near the point evaluation $\phi_\alpha^{(j)}$ in the maximal ideal space of $H_\infty[\Delta_j]$. But then the point evaluation $\phi_\lambda\in\mathfrak{M}_e[\Omega]$ is near the homomorphism ϕ in $\mathfrak{M}[\Omega]$. Thus $\mathfrak{M}_e[\Omega]$ is dense in $\mathfrak{M}[\Omega]$, and we have our result.

4. We can relax our condition on the boundary of Ω as follows. Our results are plainly invariant under conformal mapping. It is known [5, p. 377] that every finitely connected domain with no nondegenerate boundary components is conformally equivalent to a domain bounded by circles. Thus our results apply to this more general class of domains.

References

1. L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. **80** (1958), 921–930.

2. ———, *Interpolation by bounded analytic functions and the corona problem*, Ann. of Math. (2) **76** (1962), 547–559.

3. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.

4. W. Rudin, *Analytic functions of class H_p*, Trans. Amer. Math. Soc. **78** (1955), 46–66.

5. A. Hurwitz and R. Courant, *Funktionentheorie*, Springer, Berlin, 1922.

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