A NOTE ON THE JACOBI THETA FORMULA

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In this note we show that Jacobi's identity [1, p. 280]

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}t)(1+q^{2n-1}t^{-1})=\sum_{n=-\infty}^{\infty}q^{n^2}t^n \tag{1}$$

implies relations between various partition functions of two arguments, namely (5), (7) and (8) below.

In (1) take

$$q^2=xy,\qquad t=xy^{-1},\qquad |xy|<1.$$

Then (1) becomes

$$\prod_{n=1}^{\infty}(1-x^ny^n)(1+x^ny^{n-1})(1+x^{n-1}y^n)=\sum_{n=-\infty}^{\infty}x^{n(n+1)/2}y^{n(n-1)/2}. \tag{2}$$

Let $\alpha(n, m)$ denote the number of partitions of (n, m) into distinct parts

$$(a,a-1),\qquad (b-1,b)\qquad (a,b=1,2,3,\cdots),$$

so that we have the generating function

$$\sum_{n,m=0}^{\infty}\alpha(n,m)x^ny^m=\prod_{n=1}^{\infty}(1+x^ny^{n-1})(1+x^{n-1}y^n). \tag{3}$$

Then by (2) and (3)

$$\sum_{n,m=0}^{\infty}\alpha(n,m)x^ny^m=\prod_{n=1}^{\infty}(1-x^ny^n)^{-1}\sum_{r=-\infty}^{\infty}x^{r(r+1)/2}y^{r(r-1)/2}. \tag{4}$$

Since

$$\prod_{n=1}^{\infty}(1-x^ny^n)^{-1}=\sum_{n=0}^{\infty}p(n)x^ny^n,$$

where $p(n)$ is the number of unrestricted partitions of n, it follows from (4) that

$$\alpha(n,m)=p\left(n-\frac{1}{2}(n-m)(n-m+1)\right). \tag{5}$$

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It is understood in this formula that $p(-m)=0$ for $m>0$.

Thus the relation (5) is equivalent to Jacobi's identity (1).

In (2) replace x, y by $-x$, $-y$, respectively, so that

$$\prod_{n=1}^{\infty}(1 - x^n y^n)(1 - x^n y^{n-1})(1 - x^{n-1} y^n) = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(n+1)/2} y^{n(n-1)/2}. \tag{6}$$

Now let $\beta(n, m)$ denote the number of partitions of (n, m) into (not necessarily distinct) parts

$$(a, a), \ (b, b-1), \ (c-1, c) \qquad (a, b, c = 1, 2, 3, \cdots).$$

Then (6) implies

$$\sum_{r=-\infty}^{\infty} (-1)^r x^{r(r+1)/2} y^{r(r-1)/2} \sum_{n,m=0}^{\infty} \beta(n, m) x^n y^m = 1.$$

Consequently

$$\sum_{r} (-1)^r \beta\left(n - \frac{1}{2} r(r+1), m - \frac{1}{2} r(r-1)\right) = 0 \quad (n + m > 0), \tag{7}$$

where the summation is over all r such that

$$\frac{1}{2} r(r+1) \leqq n, \qquad \frac{1}{2} r(r-1) \leqq m.$$

If we put

$$\prod_{n=1}^{\infty} (1 - x^n y^{n-1})^{-1} (1 - x^{n-1} y^n)^{-1} = \sum_{n,m=0}^{\infty} \gamma(n, m) x^n y^m,$$

so that $\gamma(n, m)$ is the number of partitions of (n, m) into (not necessarily distinct) parts

$$(a, a-1), \quad (b-1, b) \qquad (a, b = 1, 2, 3, \cdots),$$

it is evident that

$$\beta(n, m) = \sum_{r=0}^{\min (n,m)} p(r) \gamma(n - r, m - r). \tag{8}$$

Reference

1. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Clarendon Press, Oxford, 1938.

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