THE CHARACTERIZATION OF FUNCTIONS ARISING AS POTENTIALS. II

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1. Statement of result. We continue our study of the function spaces L^p_α, begun in [7]. We recall that $f\in L^p_\alpha(E_n)$ when $f=K_\alpha * \phi$, where $\phi\in L^p(E_n)$. K_α is the Bessel kernel, characterized by its Fourier transform $K_\alpha(x)\hat{} = (1+|x|^2)^{-\alpha/2}$. It should also be recalled that the space L^p_k, $1<p<\infty$, with k a positive integer, coincides with the space of functions which together with their derivatives up to and including order k belong to L^p; (see [2]).

It will be convenient to give the functions in L^p_α their strict definition. Thus we redefine them to have the value $(K_\alpha * \phi)(x)$ at every *point* where this convolution converges absolutely. With this done, and if $\alpha-(n-m)/p>0$, then the restriction of an $f\in L^p_\alpha(E_n)$ to a fixed m-dimensional linear variety in E_n is well-defined (that is, it exists almost everywhere with respect to m-dimensional Euclidean measure). The problem that arises is of characterizing such restrictions.

The problem was previously solved in the following cases:

(i) When p is arbitrary, but $\alpha=1$, in Gagliardo [3].

(ii) When $p=2$, and α is otherwise arbitrary in Aronszajn and Smith [1]. In each case the solution may be expressed in terms of another function space, W^p_α, which consists of those $f\in L^p(E_n)$ for which the norm[2]

$$\|f\|_p + \left[\int_{E_n}\int_{E_n} \frac{|f(x-y)-f(x)|^p}{|y|^{n+\alpha p}}\, dxdy\right]^{1/p}$$

is finite, when $0<\alpha<1$. When $0<\alpha<2$, there is a similar definition of W^p_α (consistent with the previous one for $0<\alpha<1$) which replaces the difference $f(x-y)-f(x)$ by the second difference $f(x-y)+f(x+y)-2f(x)$. Finally for general $\alpha\geqq 2$, the spaces W^p_α are defined recurrently by $f\in W^p_\alpha$ when $f\in L^p$ and $\partial f/\partial x_n\in W^p_{\alpha-1}$, $k=1,\cdots,n$.

In stating our result we let E_m denote a fixed proper m dimensional subspace of E_n, and Rf denote the restriction to E_m of a function defined on E_n.

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[2] Such norms were considered when $n=1$ in [5]. The space is also considered in [6] and [9]; in the latter it is denoted by $\Lambda^{p,p}_\alpha$.

THEOREM. (a) *The restriction mapping R is continuous from* $L^p_\alpha(E_n)$ *to* $W^p_\beta(E_m)$, *if* $\beta=\alpha-(m-n)/p$, *as long as* $\beta>0$, *and* $1<p<\infty$.

(b) *Conversely, there exists a linear extension mapping* $\mathcal{E}$, *defined on functions of* E_m *to function of* E_n, *so that* $\mathcal{E}$ *is continuous from* $W^p_\beta(E_m)$ *to* $L^p_\alpha(E_n)$, *and* $R(\mathcal{E}(g))=g$ *for every* $g\in W^p_\beta(E_m)$, *as long as* $\beta>0$ *and* $1<p<\infty$.[3]

It should be pointed out that the spaces L^p_α, when either α is integral or $p=2$, are in some sense exceptional. Only in these cases can the elements of L^p_α be characterized in terms of the L^p modulus of continuity (i.e. conditions bearing on $\|f(x-y)-f(x)\|_p$ when say $0<\alpha\leqq 1$). In particular, L^p_α is equivalent with W^p_α only if $p=2$; see Taibleson [9]. It is known[4] that the restrictions of $W^p_\alpha(E_n)$ are elements of $W^p_\beta(E_m)$ with $\beta=\alpha-(n-m)/p$. As we shall see, this result is an immediate consequence of our theorem. Thus we have the interesting situation of two different spaces, L^p_α and W^p_α, having identical restriction spaces.

2. **Proof of the Theorem.** What follows is a sketch of the proof, details omitted. We consider the case $m=n-1$, $0<\alpha<1$; the general case is dealt with similarly. We shall make consistent use of the following notation: latin letters, $x, y, z, \cdots$ will stand for variables of E_{n-1} considered as a subspace of E_n; greek letters $\xi, \eta, \zeta, \cdots$ for points in E_1, which is the orthogonal subspace. Thus the pair(x,ξ) belongs to E_n. Also if $f(x,\xi)$ is a function defined on E_n, then $\|f(\cdot,\xi)\|_p$ will denote L^p norm with respect to the x variable, ξ fixed; $\|f(\cdot,\cdot)\|_p$ will denote the norm taken over both variables. Using the same convention, $\|g(\cdot+y)-g(\cdot)\|_p$ will stand for

$$\left(\int_{-\infty}^{+\infty} | g(x+y)-g(x)|^p dx\right)^{1/p}.$$

We make consistent use of the following classical estimate [4].

LEMMA. *If* $\Phi(\xi)=\int_0^\infty K(\xi,\eta)\phi(\eta)d\eta$, *where K is homogeneous of degree* -1, *then* $\int_0^\infty|\Phi(\xi)|^p d\xi\leqq A^p\int_0^\infty|\phi(\eta)|^p d\eta$, *where*

$$A=\int_0^\infty |K(1,\eta)|\,\eta^{-1/p}d\eta<\infty.$$

Now suppose that $f\in L^p_\alpha(E_n)$; then $f=K_\alpha*\phi$ where $\phi\in L^p(E_n)$; and the norm of f in L^p, $\|f\|_{p,\alpha}$, is given by $\|f\|_{p,\alpha}=\|\phi\|_p$. Let $g=R(f)$. Then

[3] The mapping $\mathcal{E}$ is defined on all locally integrable functions of E_m.

[4] This result is due to several Soviet authors. For references see [9], and the paper of O. V. Besov in Trudy Steklov Inst. Acad. Sci. USSR 60 (1961), 42–81.

$$g(x) = \int_{-\infty}^{\infty}\int_{E_{n-1}} \phi\,(x - z, \xi)K_\alpha(z, \xi)dzd\xi.$$

Hence,

$$|\,g(x)\,| \leqq \int_{E_{n-1}} \|\phi(x - z, \cdot)\|_p \|K(z, \cdot)\|_q dz,$$

where $1/p+1/q=1$. From this it follows that

$$\|g\|_p \leqq \|\phi(\cdot, \cdot)\|_p \int_{E_{n-1}} \|K(z, \cdot)\|_q dz = A\|\phi\|_p = A\|f\|_{p,\alpha}. \tag{1}$$

This is a consequence of the fact that $\int_{E_{n-1}}\|K_\alpha(z, \cdot)\|_q dz < \infty$ if $\alpha-1/p>0$, which follows easily from the estimates

$$K_\alpha(z, \xi) = O(|\,z\,|^2 + \xi^2)^{(-n+\alpha)/2} \qquad \text{for } |\,z\,|^2 + \xi^2 \to 0,$$

and

$$K_\alpha(z, \xi) = O\left(\exp - \frac{(|\,z\,|^2 + \xi^2)^{1/2}}{2}\right) \qquad \text{for } |\,z\,|^2 + \xi^2 \to \infty;$$

see [1].

Next, define $g_\xi(x)$ by $g_\xi(x)=\int_{E_{n-1}}\phi(x-z, \xi)K_\alpha(z, \xi)dz$. Thus $g(x) =\int g_\xi(x)d\xi$. We have

$$\|g_\xi(\cdot + y) - g_\xi(\cdot)\|_p \leqq \|\phi(\cdot, \xi)\|_p \int_{E_{n-1}} |\,K_\alpha(z - y, \xi) - K_\alpha(z, \xi)\,|\,dz.$$

Using the fact, (see, [1]) that $\nabla K_\alpha=O(|x|^2+\xi^2)^{(-n-1+\alpha)/2}$ and the previous estimates on K_α, it can be shown that the last integral is dominated by $A|y|^{-1+\alpha}\psi(\xi/|y|)$, where $\psi(u)=O(|u|^{-1+\alpha})$ as $u\to 0$ and $O(|u|^{-2+\alpha})$ as $u\to\infty$, $(0<\alpha<1$, here). From this it follows that

$$\|g(\cdot - y) - g(\cdot)\|_p$$
$$\leqq A\left[\int_{|\xi|\leq|y|} |\,\xi\,|^{-1+\alpha}\|\phi(\cdot, \xi)\|_p d\xi + |\,y\,| \int_{|\xi|\geq|y|} |\,\xi\,|^{-2+\alpha}\|\phi(\cdot, \xi)\|_p d\xi\right].$$

An application of the lemma then shows, since $1>\alpha>1/p$,

$$\int_{E_{n-1}} \frac{\|g(\cdot - y) - g(\cdot)\|_p^p}{|\,y\,|^{n-2+\alpha p}} dy \leqq A\int_{-\infty}^{\infty}\|\phi(\cdot, \xi)\|_p^p d\xi \tag{2}$$
$$= A\|\phi\|_p^p = A\|f\|_{p,\alpha}^p.$$

Combining this with (1) above proves part (a) of the theorem. To prove the converse, assume that $g \in W^p_\beta(E_{n-1})$, and g is sufficiently smooth. The smoothness is no restriction of generality since our estimates will be seen to be uniform in the norm. To define the extension operator, choose $\psi \in C_0^\infty(E_{n-1})$, $\int_{E_{n-1}} \psi(x)dx = 1$, and ψ vanishes outside the unit sphere. Also choose $\lambda \in C_0^\infty(E_1)$ so that $\lambda(0) = 1$.

Let

$$\mathcal{E}(g) = f(x, \xi) = \lambda(\xi) \,|\, \xi \,|^{-n+1} \int_{E_{n-1}} g(x - y)\psi(y/\,|\,\xi\,|\,)dy. \tag{3}$$

Notice that $f(x, 0) = g(x)$ and $\|f(\cdot, \cdot)\|_p \leqq A\|g\|_p$.

In order to prove that $f \in L^p_\alpha(E_n)$, we shall consider $F = J_{1-\alpha}(f)$, and show that $F \in L^p_1(E_n)$. This will suffice because $J_{1-\alpha}$ is a norm-preserving isomorphism of L^p_α onto L^p_1. To prove $F(x, \xi) \in L^p_1(E_n)$ it suffices to show that F, $\partial F/\partial x_k$, $\partial F/\partial \xi$ all belong to $L^p(E_n)$. However, this is clear for F itself, because $f \in L^p(E_n)$ and $J_{1-\alpha}$ does not increase the L^p norm. Thus we consider $\partial F/\partial x_k$. Now

$$F(x, \xi) = \int\int K_{1-\alpha}(z, \eta)f(x - z), \xi - \eta)dzd\eta.$$

However

$$\left| \frac{\partial f}{\partial x_k}(x, \xi) \right| \leqq A \,|\, \xi \,|^{-n} \int_{|y| \leq |\xi|} |\, g(x - y) - g(x) \,|\, dy,$$

by (3), because

$$\int \frac{\partial}{\partial x_k} \psi(x)dx = 0$$

and ψ vanishes outside the unit sphere. Also, as we have seen $|K_{1-\alpha}(z, \xi)| \leqq A(|z|^2 + \xi^2)^{(-n+1-\alpha)/2}$. Therefore we see

$$\begin{aligned} \left| \frac{\partial}{\partial x_k} F(x, \xi) \right| \leqq A \int\int (\,|\, z \,|^2 + \eta^2)^{(-n+1-\alpha)/2} \,|\, \xi - \eta \,|^{-n} \\ \cdot \int_{|y| \leq |\xi - \eta|} |\, g(x - y - z) - g(x - z) \,|\, dydzd\eta. \end{aligned} \tag{4}$$

Let us now set $\omega(y) = \|g(\cdot - y) - g(\cdot)\|_{p}$,) and

$$\Omega(\rho) = \rho^{-n+1-\alpha} \int_{|y| < \rho} \omega(y)dy, \qquad 0 < \rho < \infty.$$

Then by (4) and Minkowski's inequality for integrals we get

$$\left\| \frac{\partial F}{\partial x_k}(\cdot, \xi) \right\|_p$$

$$\leqq A \int_{E_{n-1}} \int_{-\infty}^{\infty} (|z|^2 + \eta^2)^{(-n+1-\alpha)/2} |\xi - \eta|^{-1+\alpha} \Omega(|\xi - \eta|) d\eta dz.$$

Carrying out the integration of z over E_{n-1} gives

$$\left\| \frac{\partial F}{\partial x_k}(\cdot, \xi) \right\|_p \leqq A \int_{-\infty}^{+\infty} |\eta|^{-\alpha} |\xi - \eta|^{-1+\alpha} \Omega(|\xi - \eta|) d\eta$$

$$= A \int_{-\infty}^{\infty} |\xi - \eta|^{-\alpha} |\eta|^{-1+\alpha} \Omega(|\eta|) d\eta.$$

A two-fold application of the lemma then shows, since $\alpha > 1/p$,

$$\left\| \frac{\partial F}{\partial x_k} \right\|_p = \left\| \frac{\partial F}{\partial x_k}(\cdot, \cdot) \right\|_p \leqq A \int_0^{\infty} [\Omega(\rho)]^p d\rho \leqq A \int_{E_{n-1}} \frac{\omega^p(y) dy}{|y|^{n-1+\beta p}}$$

$$= A \int_{E_{n-1}} \int_{E_{n-1}} \frac{|g(x-y) - g(x)|^p}{|y|^{n-1+\beta_p}} dy dx.[5]$$

Similar estimates hold for $\partial F/\partial \xi$. This completes the proof of the theorem.

3. **Further remarks.** We have the following corollary of our theorem

COROLLARY. (a) *If* $f \in L^p_\alpha(E_n)$, *then* $Rf \in L^p_\beta(E_m)$, *when* $\beta = \alpha - (n-m)/p > 0$, *and* $1 < p \leqq 2$.

(b) *If* $g \in L^p_\beta(E_m)$, $\beta = \alpha - (n-m)/p > 0$, *then* $\mathcal{E}(g) \in L^p_\alpha(E_n)$, *if* $2 \leqq p < \infty$.

Part (a) of the corollary is due to Calderon [2]. Part (b) is its appropriate converse. The corollary follows from the theorem and the known continuous inclusion relations $W^p_\alpha \subset L^p_\alpha$, if $1 \leqq p \leqq 2$, and $L^p_\alpha \subset W^p_\alpha$ if $2 \leqq p \leqq \infty$; see Taibleson [9].

We shall now point out how to obtain an analogue of our theorem which replaces $L^p_\alpha(E_n)$ by $W^p_\alpha(E_n)$. Thus let $f \in W^p_\alpha(E_n)$. By part (b) of the theorem it has an extension to a function in E_{n+1} which belongs to $L^p_{\alpha+1/p}(E_{n+1})$. By part (a) this extension, when restricted to E_m, belongs to $W^p_\beta(E_m)$, where $\beta = \alpha - (n-m)/p$. However this restriction is obviously the restriction of our original f. Therefore the restriction of

[5] To prove this one may also use an n-dimensional variant of the lemma; see [8, Lemma (3.5)].

an $f \in W^p_\alpha(E_n)$ belongs to $W^p_\beta(E_m)$. In the same way the analogous extension property is proved.

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