results for operators not assumed positive by means of a reduction procedure [4] and the present theorems.

We are indebted to the work of Eberhard Hopf for suggesting that a resolution of this type is possible.

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TCHEBYCHEFF QUADRATURE IS POSSIBLE ON THE INFINITE INTERVAL[1]

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The purpose of this announcement is to state a theorem on Tchebycheff quadrature which answers a question posed in [1], and to discuss the proof. Complete details will appear elsewhere.

1. Tchebycheff quadrature.

Definition 1.1. A unit mass distribution on $(-\infty, \infty)$ possessing moments of all positive integer order will be said to belong to class D.

Definition 1.2. Let ψ be an element of D and n a positive integer. We refer to the equations

$$\frac{1}{n}\sum_{i=1}^{n} x_{i,n}^{k} = \int x^k d\psi, \qquad k = 1, \cdots, n$$

as the equations (ψ, n). These equations admit a solution $x_{1,n}, \cdots, x_{n,n}$ which is unique up to permutation of the first index.

Definition 1.3. T quadrature is said to be possible for an element ψ of D if equations (ψ, n) have real solutions for every positive integer n. If T quadrature is possible for ψ it is called a T distribution.

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LEMMA 1.1 [1]. *The mass set of a T distribution lies on a finite interval.*

DEFINITION 1.4. T_1 quadrature is said to be possible for an element ψ of D if the equations (ψ, n) do not have real solutions for every positive integer n, but do have real solutions for an infinite number of positive integers. If T_1 quadrature is possible for ψ it is called a T_1 distribution. The values of n for which equations (ψ, n) have real solutions are called the T set of ψ. If either T or T_1 quadrature is possible for ψ in D, we say Tchebycheff quadrature is possible.

We are now led to the question raised in [1], namely, is there a T_1 distribution whose mass does not lie on a finite interval, or in other words, is Tchebycheff quadrature possible on the infinite interval? Evidence is produced there that this is not so, since it is shown that if a T_1 distribution exists whose mass does not lie on a finite interval, its T set would have very large gaps.

THEOREM 1.1. *There is a T_1 distribution whose mass does not lie on a finite interval.*

2. **Discussion of proof.** Lemmas are stated here but not proved. Comments are added which will indicate how the theorem is proved.

DEFINITION 2.1. A simple distribution of degree n is a unit mass distribution consisting of equal masses at n distinct points.

DEFINITION 2.2. Let ψ, ψ' be two elements of D, and let m_k, m_k' denote the moments $\int x^k d\psi$, $\int x^k d\psi'$, respectively, $k=1, \cdots$. Let n be a positive integer. Then

$$\|\psi - \psi'\|_n$$

is defined as

$$\max\{ |m_1 - m_1'|, \cdots, |m_n - m_n'| \}.$$

LEMMA 2.1. *Let ψ be a simple distribution of degree n. There is a number $\epsilon > 0$, called a proximity number of ψ, such that if*

$$\|\psi - \psi'\|_n \leqq \epsilon,$$

where ψ' is any element of D, then the equations (ψ', n) have n distinct real solutions.

LEMMA 2.2. *There is an element ψ of D whose mass is not contained in a finite interval, and an infinite sequence of simple distributions ψ_k of degree n_k and with proximity numbers ϵ_k, $k=1, \cdots$, where the n_k tend to infinity, such that*

$$\|\psi - \psi_k\|_{n_k} \leqq \epsilon_k, \qquad k = 1, \cdots. \tag{2.1}$$

COMMENT 1. The condition (2.1) implies that equations (ψ, n_k) have real solutions for $k=1, \cdots$, so that ψ is a T_1 distribution.

LEMMA 2.3. *Let* $\{0_i\}$, $i=1, \cdots$, *be a family of nonoverlapping, finite intervals on the real axis whose union does not lie in a finite interval. There is a sequence of simple distributions* ψ_k *of degree* n_k *and proximity numbers* ϵ_k, $k=1, \cdots$, *where* n_k *tends to infinity, such that*

$$\int_{0_k} d\psi_{k+p} \geqq \gamma_k > 0, \qquad k = 1, \cdots, p = 0, \cdots, \tag{2.2}$$

and

$$\|\psi_{k+p} - \psi_k\|_{n_k} \leqq \epsilon_k, \qquad k = 1, \cdots, p = 1, \cdots. \tag{2.3}$$

COMMENT 2. From the ψ_k we can extract a sequence whose limit ψ is in D. This distribution ψ and the ψ_k of this lemma satisfy the conditions of Lemma 2.2. The mass of ψ is not on a finite interval because of (2.2), and (2.3) leads to (2.1).

COMMENT 3. In constructing the sequence ψ_k we proceed in a stepwise fashion, constructing ψ_{k+1} from ψ_k in two stages. ψ_k has all its mass on the sets $0_1, \cdots, 0_k$. We move some mass from 0_k to 0_{k+1}, thus creating a mass distribution ψ_k'. We then split each mass of ψ_k' into a number of equal masses, locating them close to the mass in which they originated. This can be done so that ψ_{k+1} is simple and has all its mass on $0_1, \cdots, 0_{k+1}$.

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