TWO APPLICATIONS OF THE METHOD OF CONSTRUCTION BY ULTRAPOWERS TO ANALYSIS

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1. **Introduction.** Recently, A. Robinson in [1] has given a proper extension of classical analysis, which he called nonstandard analysis. His theory is based on the general metamathematical result that there exist nonstandard models for the system R of real numbers. Such models of R may be constructed in the form of ultrapowers as defined by T. Frayne, D. Scott, and A. Tarski in [2]. The object of this paper is to apply Robinson's method in order to obtain a new proof of the Hahn-Banach extension theorem and in order to give a new and simple proof of a result about the existence of certain measures on Boolean algebras which was recently obtained by O. Nikodým in [3; 4].

It may be of interest to the reader to point out that the use of nonstandard arguments in the proof of the Hahn-Banach extension theorem eliminates the use of Zorn's lemma. In fact, the validity of the Hahn-Banach extension theorem is a consequence of the apparently weaker hypothesis that every proper filter is contained in an ultrafilter, i.e., the prime ideal theorem for Boolean algebras. It seems likely, that conversely the Hahn-Banach extension theorem implies the prime ideal theorem for Boolean algebras.

A more detailed presentation of the subject of this announcement will be contained in lecture notes on nonstandard analysis under preparation by the author.

2. **Nonstandard models of R.** Let R denote the real number system. Let D be an arbitrary set and let $\mathfrak{U}$ be an ultrafilter on D. If A and B are two mappings of D into R, i.e., $A, B \in D^R$, then we say that $A \equiv_{\mathfrak{U}} B$ if and only if $\{n: n \in D \text{ and } A(n) = B(n)\} \in \mathfrak{U}$. The relation $A \equiv_{\mathfrak{U}} B$ is easily seen to be an equivalence relation. The set $D^R/\mathfrak{U}$ of all equivalence classes will be denoted by R^* and the equivalence class of a mapping A of D into R will be denoted by a. Thus $A \in a$. Finally, we define the algebraic operations in R^* as follows: $a+b=c$ if and only if there exist elements $A \in a$, $B \in b$ and $C \in c$ such that $\{n: n \in D \text{ and } A(n)+B(n)=C(n)\} \in \mathfrak{U}$; and a similar definition

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for $ab=c$ and $a \leqq b$. With these definitions R^* is a totally-ordered field and $R \subseteq R^*$. If $R \neq R^*$, then R^* is non-archimedean and is a non-standard model of R. In this case the following two subsets are introduced. M_0 is the set of all $a \in R^*$ such that $|a| < r$ for some $r \in R$. Then M_0 is a ring and the elements of M_0 are called the finite elements of R^*. M_1 is the set of all $a \in R^*$ such that $|a| < r$ for all $r \in R$ and $r > 0$. The elements of M_1 are called infinitesimals. Furthermore, M_1 is a maximal ideal of M_0 and M_0/M_1 is isomorphic to R. The homomorphism of M_0 onto R with kernel M_1 will be called "standard part" and will be denoted by st. If $a \in M_0$, then st(a) is the unique real number which is infinitely close to a. This homomorphism is order preserving.

The terminology used in this section is taken from [1].

3. **The Hahn-Banach extension theorem.** In this section we shall sketch a proof of the Hahn-Banach extension theorem using non-standard arguments.

THEOREM (HAHN-BANACH). *Let E be a real linear space and let p be a sublinear functional defined on E, i.e., a mapping p of E into R such that $p(x+y) \leqq p(x)+p(y)$ for all $x, y \in E$ and $p(tx)=tp(x)$ for all $x \in E$ and all real $t \geqq 0$. If f is a real linear functional defined on a linear subspace G of E such that $f(x) \leqq p(x)$ for all $x \in G$, then there exists a real linear functional F on E such that $F(x)=f(x)$ for all $x \in G$ and $F(x) \leqq p(x)$ for all $x \in E$.*

PROOF. Let $\{f_n : n \in D\}$ be the family of all linear functionals which are defined on some linear subspace of E which contains G and which have the following properties: $f_n(x)=f(x)$ for all $x \in G$ and $f_n(x) \leqq p(x)$ for all $x \in E$ for which $f_n(x)$ is defined. It is evident that $D \neq \emptyset$. For every $x \in E$ we denote by D_x the set of all indices $n \in D$ such that the domain of f_n contains x. It follows from Banach's proof (see [5, p. 28]) that $D_x \neq \emptyset$ for all $x \in E$. Furthermore, the family $\{D_x : x \in E\}$ of subsets of D has the finite intersection property, i.e., if $x_1, \cdots, x_n$ are elements of E, then $\bigcap_{i=1}^n D_{x_i} \neq \emptyset$. Indeed, apply Banach's construction successively to the elements $x_1, \cdots, x_n$. Hence, there exists an ultrafilter $\mathfrak{U}$ on D which contains the family $\{D_x : x \in E\}$. Let R^* be the ultrapower $D^R/\mathfrak{U}$. Then we define the following mapping $\bar{f}$ of E into R^*. If $x \in E$, then $\bar{f}(x)$ is that element of R^* which is determined by an element A of D^R such that $A(n)=f_n(x)$ for all $n \in D_x$. Then it is easy to see that $\bar{f}$ is a linear transformation of E into R^* (consider R^* as a vector space over R) and that $\bar{f}$ has the following properties: (i) $\bar{f}(x)=f(x)$ for all $x \in G$ and (ii) $\bar{f}(x) \leqq p(x)$ for all $x \in E$.

From (ii) it follows that $-p(-x) \leqq \tilde{f}(x) \leqq p(x)$ for all $x \in E$, i.e., $\tilde{f}(x)$ is finite for all $x \in E$. Hence, $F(x) = \mathrm{st}(\tilde{f}(x))$ is the required linear functional. This completes the proof of the theorem.

REMARK. The proof shows that the ultrafilter $\mathfrak{U}$ is fixed, i.e., there exists an element $n \in D$ such that $\{n\} \in U$. Furthermore, there exists a one-to-one correspondence between the family of all ultrafilters on D containing the family $\{D_x: x \in E\}$ and the family of all extensions of f satisfying the conditions of the theorem.

4. A theorem of Nikodým. Let B be a Boolean algebra. It is well-known that there does not always exist on B a strictly positive real-valued finitely additive measure. Therefore, the following result, which was recently obtained by O. Nikodým in [**3**; **4**], is of interest.

THEOREM (O. NIKODYM). *For every Boolean algebra B there exists a totally ordered field F which is in general non-archimedean such that B admits a strictly positive F-valued finitely additive measure.*

PROOF. Let B be a Boolean algebra and let $\{\mu_n: n \in D\}$ be the collection of all real-valued measures on B such that $\mu_n(1) = 1$ for all $n \in D$. For every $0 \neq a \in B$ we denote by D_a the set of all $n \in D$ such that $\mu_n(a) \neq 0$. It is well known that $D \neq \emptyset$ for all $0 \neq a \in B$ (Stone's Theorem). Hence, the family of sets $\{D_a: 0 \neq a \in B\}$ has the finite intersection property. Let $\mathfrak{U}$ be an ultrafilter on D which contains the family $\{D_a: 0 \neq a \in B\}$. Let F be the ultrapower $D^R/\mathfrak{U}$. Then F is a totally-ordered field, $R \subseteq F$ and $R \neq F$ if and only if F is non-archimedean. We define now the following mapping $\tilde{\mu}$ of B into F. If $0 \neq a \in B$, then $\tilde{\mu}(a)$ is that element of F which is determined by the element $A \in D^R$ which has the following property: $A(n) = \mu_n(a)$ for all $n \in D_a$; and we define $\tilde{\mu}(0) = 0$. Then, by construction, $\tilde{\mu}$ has the following properties: (i) $\tilde{\mu}(a) = 0$ if and only if $a = 0$, i.e., $\tilde{\mu}$ is strictly positive and (ii) $\tilde{\mu}(a \vee b) = \tilde{\mu}(a) + \tilde{\mu}(b)$ whenever $a \wedge b = 0$, i.e., $\tilde{\mu}$ is finitely-additive. This completes the proof of the theorem.

REMARK. If B does not admit a strictly positive real-valued finitely additive measure, then the totally-ordered field F constructed in the proof of the preceding theorem is a proper extension of R and hence, $\tilde{\mu}(a)$ is infinitesimal for at least one element $0 \neq a \in B$.

REFERENCES

A. ROBINSON

1. *Non-standard analysis*, Nederl. Akad. Wetensch. Proc. Ser. A **64** (1961) = Indag. Math. **23** (1961), 432–440.

T. E. FRAYNE, D. S. SCOTT AND A. TARSKI

2. *Reduced products*, Notices Amer. Math. Soc. **5** (1958), 673.

O. NIKODÝM

3. *On extension of a given finitely additive field-valued, non-negative measure, on a finitely additive Boolean tribe, to another tribe more ample*, Rend. Sem. Mat. Univ. Padova **26** (1956), 232–327.
4. *Sur la mesure non archimédienne effective sur une tribu de Boole arbitraire*, C. R. Acad. Sci. Paris **251** (1960), 2113–2115.

S. BANACH

5. *Théorie des opérations linéaires*, Warszawa, 1932.

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