A NOTE ON THE KO-THEORY OF SPHERE-BUNDLES[1]

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1. Two vector bundles E and F over the finite connected CW complex X are J-equivalent, if their sphere-bundles $S(E)$ and $S(F)$ are of the same fiber-homotopy type. If they become J-equivalent after a suitable number of trivial bundles is added to both of them they are stably J-equivalent.

This note concerns itself with a new stable J-invariant $\theta(E)$, which was suggested by the recent work of Atiyah-Hirzebruch [**2**] and F. Adams [**1**]. In fact $\theta(E)$ bears the same relation to the Adams operation ψ_i, as the Stiefel-Whitney class—a known J-invariant—bears to the Steenrod operations.

2. We assume familiarity with the Grothendieck group, $KO(X)$, of real vector bundles over X, and with the extension of this functor to a cohomology theory: $KO^*(X) = \sum_{i=+\infty}^{i=-\infty} KO^i(X)$; $KO^0(X) = KO(X)$. Also that the exterior powers,

$$\lambda^i\colon KO(X) \to KO(X)$$

as defined in [**3**] are understood. In terms of these the Adams operations ψ_i, are defined as follows:

$$\text{Set } \lambda_t(x) = \sum \lambda^i(x)t^i, \quad x \in KO(X), \quad \lambda_t(x) \in KO(X)[[t]].$$

Now define $\psi_{-t}(x)$ as the power series:

$$(2.1) \qquad \psi_{-t}(x) = -\, td/dt\{\log \lambda_t(x)\} = -\, t\lambda_t'(x)/\lambda_t(x)$$

and set $\psi_i(x)$ equal to the coefficient of t^i in $\psi_t(x)$.

The familiar formula: $\lambda_t(x+y) = \lambda_t(x)\cdot\lambda_t(y)$ then goes over into $\psi_t(x+y) = \psi_t(x) + \psi_t(y)$ so that the ψ_i are additive and therefore much more tractable than the λ^i. Note that if L is a line-bundle then by (2.1) $\psi_k L = L^k$. Other important properties of ψ_k are [**1**]:

(2.2) ψ_k is a ring homomorphism, which commutes with the λ^i.

$$(2.3) \qquad \psi_k \cdot \psi_s = \psi_{ks}.$$

3. In the KO^* terminology the periodicity theorem for the orthogonal groups [**4**; **2**] states that the map

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$$(3.1) \qquad KO^*(X) \otimes KO(S^{8n}) \to KO^*(X \times S^{8n})$$

induced by the tensor product of bundles is an isomorphism. The following theorem, which is implicitly contained in [2], extends this result:

THEOREM I. *Let E have dimension $(8n+1)$ over X, and assume that the structure group of E may be reduced to* $\mathrm{Spin}(8n+1)$. *Let π: $S(E) \to X$ be the associated sphere-bundle and write $\hat{E}$ for the bundle along the fibers in $S(E)$. The structure group of $\hat{E}$ then has a reduction to* $\mathrm{Spin}(8n)$, *and we let $\Delta^+(\hat{E})$ be the vector bundle associated to $\hat{E}$ by one of the real* Spin-*representations of* $\mathrm{Spin}(8n)$.

Under these circumstances $KO^*\{S(E)\}$ is a free module over $KO^*(X)$ with generators, 1 *and* $\Delta^+(\hat{E})$:

$$(3.2) \qquad KO^*\{S(E)\} = KO^*(X)[y], \qquad y = \Delta^+(\hat{E}).$$

An immediate consequence of Theorem I is that formulae of the type:

$$(3.3) \qquad \begin{aligned} y^2 &= A(E)y + B(E) \\ \psi_k y &= \theta_k(E)y + \Gamma_k(E) \end{aligned} \qquad y = \Delta^+(\hat{E}).$$

must hold, and thus define four invariants of E in $KO(X)$. The $\theta(E)$ will be fashioned from the $\theta_k(E)$.

Remark first that a fiber homotopy equivalence f, between $S(E)$ and $S(F)$ induces a $KO^*(X)$-homomorphism, f^* of $KO^*\{S(F)\}$ onto $KO^*\{S(E)\}$.

Hence:

THEOREM II. *The bundles E and E' subject to the condition of Theorem* I *are J-equivalent only if*:

$$(3.4) \qquad \theta_k(E) = \theta_k(E')\cdot\psi_k u/u, \qquad k \in Z^+$$

where $u \in KO(X)$ is an invertible element, i.e. $\dim u = 1$.

We have in addition: (Adams [1], see also (4.7)).

PROPOSITION 3.1. *When E is the trivial bundle, then*

$$(3.5) \qquad \theta_k(E) = k^{4n}.$$

One may now clearly formalize the stable instance of (3.4) and (3.5) as follows:

DEFINITION. A function f: $Z^+ \to KO(X)$ is called a cocycle if

$$(3.6) \qquad f(ts) = \{\psi_t f(s)\} f(t), \qquad s, t \in Z^+,$$

$$(3.7) \qquad \dim f(t) = t^{4n(f)},$$

where $n(f)$ is a fixed integer $\in Z^+$.

The cocycles form a multiplicative system, $f\cdot g(t) = f(t)\cdot g(t)$. Two cocycles f, g are called equivalent, if there exist integers n, $m \in Z^+$, and an element $u \in KO(X)$ with $\dim u = 1$, so that:

$$t^{4n}f(t) = \{t^{4m}(\psi_t u)/u\}\cdot g(t), \qquad t \in Z^+.$$

The equivalence classes are then seen to form an abelian group which is denoted by $H^1(Z^+; KO(X))$.

Consider now the bundle E of our discussion. By (3.3) and (3.5) the function $t \to \theta_t(E)$ determines a cocycle and hence an element $\theta(E) \in H^1(Z^+; KO(X))$.

THEOREM III. *The element $\theta(E)$ is a stable J-invariant of E. Further, one has $\theta(E \oplus E' \oplus 7\cdot 1) = \theta(E) + \theta(E')$, so that θ extends to a homomorphism*

$$(3.8) \qquad \theta\colon K\ \mathrm{Spin}(X) \to H^1(Z^+; KO(X)).$$

Here of course K Spin(X) is the subring of $KO(X)$ generated by vector bundles which admit a Spin-reduction.

It is customary to denote the stable J-classes of vector-bundles by $J(X)$. If we define $\Theta(X)$ as the image of $\theta\colon K\,\mathrm{Spin}(X) \to H^1(Z^+; KO(X))$, then Theorem III shows that $\Theta(X)$ furnishes a *lower bound* for $J(X)$ in the sense that θ induces a surjection of a subgroup of $J(X)$ onto $\Theta(X)$.

4. We have seen that the Theorems II and III are really quite straight forward consequences of Theorem I. The question now arises how the invariates $\theta_k(E)$, etc., are to be computed. For instance can they be expressed in terms of the $\lambda^i(E)$?

THEOREM IV. *Let E be as in Theorem* I *and set $\Delta(E) \in KO(X)$ equal to the bundle which the spin representation of* Spin$(8n+1)$ *associates to E, via the Spin reduction of E.*

Then

$$(4.1) \qquad \{\Delta(E)\}^2 = \frac{1}{2}\lambda_1(E) \qquad \textit{in } KO(X).$$

(4.2) *The invariants A, B, θ_k, Γ_k, of E are given by universal polynomials in $\Delta(E)$, and the $\lambda^i(E)$.*

Particular examples are as follows:

(4.3) Both, $A(E)$ and $\theta_2(E)$ are equal to $\Delta(E)$,

$$B(E) = -\sum_{i=1}^{2n-1} \lambda^{2i-1}(E-1), \tag{4.4}$$

$$\theta_{2k+1}(E) \text{ is a polynomial in the } \lambda^i E. \tag{4.5}$$

In general the expression for θ_k is quite complicated as the following recipe for θ_k shows:

Algorithm. Consider the ring of finite Laurent-series $L=Z[x_i, x_i^{-1}]$, $i=1, \cdots, 4n$. Define γ^i, ω, and η_k in L by:

$$\sum_0^\infty \gamma^i t^i = (1+t)\prod_1^{4n}(1+tx_i^2)(1+tx_i^{-2}),$$

$$\omega = \prod_1^{4n}(x_i + x_i^{-1}),$$

$$\eta_k = \prod_1^{4n}(x_i^k + \cdots + x_i^{-k}).$$

Express η_k as a polynomial in the γ^i, and ω:

$$\eta_k = P_k(\gamma^i, \omega),$$

then

$$\theta_k(E) = P_k(\lambda^i(E); \Delta(E)).$$

In special circumstance θ_k can of course be computed much more simply. As an example we cite:

Proposition 4.1. *If $E=8nL+1$, where L is a line bundle, then*:

$$\theta_k(E) = (1 + L + \cdots + L^{k-1}) \tag{4.6}$$

whence

$$\theta_k(E) = \begin{cases} k^{4n} + \{k^{4n}/2\}(L-1) & k \text{ even}, \\ k^{4n} + \{(k^{4n}-1)/2\}(L-1) & k \text{ odd}. \end{cases} \tag{4.7}$$

Note that (4.6) and (4.7) imply Proposition 3.1.

The general idea behind these computations is the following one: If G is a Lie group, we write $RO(G)$ for the character ring (Grothendieck ring) of the finite dimensional G modules over R (see [2]). The exterior powers λ^i, and hence the ψ_i also may be introduced into these rings just as they were introduced into $KO(X)$. From representation theory we learn that:

Proposition 4.2. *If we write G for* Spin$(8n+1)$, *H for* Spin$(8n)$ *and let $\iota: H \to G$ be the standard inclusion. Then*

(4.8) $\iota^*: RO(G) \to RO(H)$ *is an injection,*

(4.9) $RO(H)$ *is a free module over* $RO(G)$ *generated by the unit element and the spin representation* Δ^+,

(4.10) $RO(G)$ *is a polynomial ring with generators*

$$\rho, \lambda'\rho, \cdots, \lambda^{4n}\rho, \Delta$$

where ρ *is the standard* $(8n+1)$ *dimensional* G*-module and* Δ *is the spin-representation.*

COROLLARY. *There are unique elements* $A, B, \theta_k, \Gamma_k \in RO(G)$ *such that*

$$(4.11)\qquad \begin{aligned} (\Delta^+)^2 &= \Delta^+ \otimes i^*A + i^*B, \\ \psi_k(\Delta^+) &= \Delta^+ \otimes i^*\theta_k + i^*\Gamma_k. \end{aligned}$$

Further these are well determined polynomials in $\lambda^i\rho$, Δ.

The similarity of these formulae to (3.3) is clear, and in fact the corollary yields Theorem IV by virtue of the following two quite general facts:

1. If G is a Lie group and θ an element in $RO(G)$, then θ defines a functor $\xi \to \theta(\xi)$ from principal G-bundles over X into $KO(X)$: if θ is an "actual" G-module $\theta(\xi)$ is the associated vector bundle.

2. Suppose now that ξ has total space Y_ξ and that $\iota: H \to G$ is a closed subgroup of G. Then $Y_\xi \to Y_\xi/H$ defines Y_ξ as a principal H bundle, $\hat{\xi}$, over Y_ξ/H. Consider the G/H-bundle π: $Y_\xi/H \to X$. Under these circumstances the following identity holds in $KO(Y_\xi/H)$.

Permanence Law. Let $\alpha \in RO(H)$, $\beta \in RO(G)$. Then

$$(\alpha \otimes i^*\beta)(\hat{\xi}) = \alpha(\hat{\xi}) \otimes \pi^*\beta(\xi) \qquad \text{in } KO(Y_\xi/H).$$

REMARKS. 1. The invariant θ seems to yield the best presently known information about $J(X)$. For instance with the aid of (4.3) and (4.1) one may compute $\Theta\{S_{4k}\}$ and obtains a cyclic group of order equal to the denominator of the kth Bernoulli number over $4k$. For $\Theta\{S_n\}$, $n=1$, 2 mod 8 one finds (via (4.1)) the group Z_2. Finally this same formula leads to the result that for real projective space $\tilde{K}O(P_n) \simeq J(P_n)$. This beautiful result of Adams is the central step in his solution of the vector-field problem on spheres.

2. Completely analogous formulae hold for the KU theory under more general circumstances. (E need only admit a reduction to $\text{Spin}(2n+1) \times_{Z_2} S^1$.)

3. The operation of λ^i in $KO^*\{S(E)\}$ could be determined in an

analogous fashion but lead to very messy formulae, which furthermore give no additional stable information.

4. Finally a word concerning the proof of Theorem I. It is a known result that when $X=$ point, then $KO\{S(E)\}=KO(S^{8n})$ is generated by 1 and y. (See [2]). Hence (2.1) proves the first statement of Theorem I whenever E is trivial. Now an inductive Meyer-Vietoris argument yields the general case.

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A CONNECTION BETWEEN TAUBERIAN THEOREMS AND NORMAL FUNCTIONS

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The purpose of this note is to point out that certain Tauberian theorems follow immediately from some recent research of Lehto-Virtanen and Bagemihl-Seidel.

Let D denote the open unit disk, let C denote the unit circumference, and let $\rho(z_1, z_2)$ denote the non-Euclidean hyperbolic distance between the points z_1 and z_2 in D.

THEOREM. *Suppose that* $f(z) = \sum a_n z^n$ *and that* $n|a_n| \leqq M$ $(n=1, 2, \cdots)$ *for some constant* M. *Further, suppose that* $\{z_n\}$ *is a sequence of points in* D *converging to a point* ζ *in* C *with the property that* $\rho(z_n, z_{n+1})\to 0$ *as* $n\to\infty$. *Then, if* $f(z_n)\to c$ *as* $n\to\infty$, *the series* $\sum a_n\zeta^n$ *converges to the sum* c.

PROOF. The hypothesis implies that $|f'(z)|\leqq M/(1-|z|)$. Consequently, $\rho(f(z))|dz|\leqq 2Md\sigma(z)$ holds for all z in D where $\rho(f(z)) = |f'(z)|/(1+|f(z)|^2)$ denotes the spherical derivative of f and $d\sigma(z)$

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