AN EXPANSION FORMULA FOR DIFFERENTIAL EQUATIONS

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Denote by $z=(z^1, \cdots, z^n)$ the coordinates of the n-dimensional complex space. If $m=(m_1, \cdots, m_n)$ is an n-tuple of nonnegative integers, then we write $z^m=(z^1)^{m_1} \cdots (z^n)^{m_n}$. Any polycylinder mentioned in this paper will have its center at the origin.

Let $f(t, z)$ depend on the complex variables z and a parameter t over a measurable set I in a measure space with measure μ. We say that $f(t, z)$ is dominatedly integrable over I for z in a polycylinder U, if the following conditions are satisfied:

(a) For a.e. (almost every) value of the parameter t in I, $f(t, z)$ is holomorphic in U.

(b) The expansion $\sum a_m(t)z^m$ of $f(t, z)$ is such that each coefficient a_m is integrable over I.

(c) The series $\sum \int_I |a_m(t)| d\mu z^m$ converges in U.

It can be easily shown that $F(z)=\int_I f(t, z)d\mu$ is holomorphic in U, and this integration commutes with partial differentiation with respect to z.

DEFINITION. We say that $A(t)=\sum a^i(t, z)\partial/\partial z^i$, t being a real variable, is a t.d.i.t. (time dependent infinitesimal transformation) if there exists a polycylinder U such that, for z in U, each $a^i(t, z)$ is dominatedly integrable in the sense of Lebesgue over any finite interval I.

DEFINITION. For $f(z)$ holomorphic about the origin, we define $T^*(A; t, t_0)f$, or simply $T^*(t)f$, to be the sum function of the series

$$T_0^*(t)f + \cdots + T_r^*(t)f + \cdots,$$

where $T_0^*(t)f=f$ and, for $r>0$, $T_r^*(t)f=\int_{t_0}^t T_{r-1}^*(s)A(s)fds$.

Our main purpose is to prove the formula

$$(T^*(t)f)(z_0) = f(T(t)z_0), \tag{1}$$

where $z=T(t)z_0$ denotes the solution of the system of differential equations $dz^i/dt=a^i(t, z)$ with the initial condition $z(t_0)=z_0$.

By direct computation, it is verified that

$$\frac{d}{dt}\sum_{i=0}^{r} (T_i^*(t)f)(T_{r-i}^*(t)g) = dT_r^*(t)(fg)/dt.$$

Consequently, if $T^*(t)f$ and $T^*(t)g$ both converge absolutely in a neighborhood of the origin, then

$$T^*(t)(fg) = (T^*(t)f)(T^*(t)g).$$

In other words, $T^*(t)$ can be looked upon as an endomorphism of the ring of the functions holomorphic about the origin.

For $f(z) = \sum a_m z^m$ and $g(z) = \sum b_m z^m$ holomorphic about the origin, we write $f \ll g$ when $|a_m| \leqq b_m$ for all m. For two t.d.i.t. $A(t)$ and $B(t)$, we write $A(t) \ll B(t)$ when $A(t)z^i \ll B(t)z^i$, $i = 1, \cdots, n$, for a.e. value of t.

We obtain, in a rather straightforward manner, the following result:

THEOREM 1. *If $A(t) \ll B(t)$ and if $f \ll g$, then, for $t > t_0$, the existence of $T^*(B; t, t_0)g$ in a polycylinder U will imply that of $T^*(A; t, t_0)f$ in the same polycylinder. Moreover*

$$T^*(A; t, t_0)f \ll T^*(B; t, t_0)g.$$

It is known that, if X is an infinitesimal transformation (holomorphic about the origin and independent on the time t), then $T^*(X; t, t_0)f = (\exp(t - t_0)X)f$ exists. By constructing a suitable X with $A(t) \ll X$, we are led to the next proposition:

THEOREM 2. *If $|A(t)z^i| \leqq M$ for a.e. value of t and for z in a given polycylinder of radius R, then $T^*(t)f$ exists and is holomorphic about the origin provided the number $|t - t_0| M/R$ is sufficiently small.*

Hereafter we shall assume that $A(t)$ satisfies the conditions of the above theorem. For our purpose, this assumption is almost superficial, because $T^*(t_1)$ remains the same when $A(t)$ is subject to any alteration for values of t beyond the interval between t_0 and t_1.

Let z_0 be any point sufficiently close to the origin. Denote by $\alpha_{z_0}(t)$ the path given by

$$z(\alpha_{z_0}(t)) = (T^*(t)z)(z_0).$$

Since $T^*(t)$ is an endomorphism of the ring of the holomorphic functions about the origin, we obtain, for any polynomial f of z,

$$f(\alpha_{z_0}(t)) = (T^*(t)f)(z_0). \tag{2}$$

By passing to limit, the above identity also holds for any function f holomorphic about the origin. It follows that

$$\begin{aligned} d\overset{i}{z}(\alpha_{z_0}(t))/dt &= (T^*(t)A(t)z^i)(z_0) \\ &= (T^*(t)a^i(t, z))(z_0) \\ &= a^i(t, z(\alpha_{z_0}(t))). \end{aligned}$$

In short, $\alpha_{z_0}(t) = T(t)z_0$. Hence (1) follows from (2).

In order to indicate some application of the above results, we consider again the system of differential equations $dz^i/dt = a^i(t, z)$ represented by the t.d.i.t. $A(t)$. Denote by $a_r^i(t, z)$ the component of degree r in the expansion of the function $a^i(t, z)$ about the origin, and write $A_r(t) = \sum_{i=1}^n a_r^i(t, z)\partial/\partial z^i$. Then we study local properties of the solutions of the system of differential equations through the power series expansion of $T^*(t)z$ in z.

EXAMPLE. Consider the system of the differential equations

$$\begin{aligned} dx/dt &= -y + f(x, y), \\ dy/dt &= x + g(x, y), \end{aligned} \tag{3}$$

where f and g are holomorphic with their respective expansions of order at least 2 about $(x, y) = (0, 0)$. Set $z = e^{-it}(x + iy)$ and $\bar{z} = e^{it}(x - iy)$. Then

$$\begin{aligned} dz/dt &= e^{-it}h(e^{it}z, e^{-it}\bar{z}), \\ d\bar{z}/dt &= e^{it}\bar{h}(e^{-it}\bar{z}, e^{it}z), \end{aligned} \tag{4}$$

where $h(x+iy, x-iy) = f(x, y) + ig(x, y)$, and $\bar{h}$ is obtained from h by replacing, in the power series expansion of h, each coefficient by the conjugate. Corresponding to (4), we have the t.d.i.t.

$$A(t) = e^{-it}h(e^{it}z, e^{-it}\bar{z})\partial/\partial z + e^{it}h(e^{-it}\bar{z}, e^{it}z)\partial/\partial\bar{z}.$$

It is clear that $A_0(t) = A_1(t) = 0$ and $\int_0^{2\pi} A_2(t)dt = 0$. The first nonvanishing component of the power series expansion of $T^*(A; 2\pi, 0)z - z$ has degree 3 and is equal to

$$\left\{ \int_0^{2\pi} A_3(s)ds + \int_0^{2\pi}\int_0^s A_2(s)A_2(s')dsds' \right\} z.$$

Write

$$\begin{aligned} A_3(t) = {} & e^{-it}(\cdots + ke^{it}z^2\bar{z} + \cdots)\partial/\partial z \\ & + e^{it}(\cdots + \bar{k}e^{-it}z\bar{z}^2 + \cdots)\partial/\partial z. \end{aligned}$$

Then $\int_0^{2\pi} A_3(s)dsz = 2\pi kz^2\bar{z}$. On the other hand, we write

$$\begin{aligned} A_2(t) = {} & e^{-it}(ae^{2it}z^2 + 2bz\bar{z} + ce^{-2it}\bar{z}^2)\partial/\partial z \\ & + e^{it}(\bar{a}e^{-2it}\bar{z}^2 + 2\bar{b}z\bar{z} + \bar{c}e^{2it}z^2)\partial/\partial\bar{z}, \end{aligned}$$

and obtain

$$\int_0^{2\pi}\int_0^s A_2(s)A_2(s')dsds'z = -4\pi iz^2\bar{z}(2b\bar{b} - ab + c\bar{c}/3).$$

Therefore

$$T^*(A;2\pi,0)z = z + 2\pi z^2\bar{z}(k - 2i(2b\bar{b} - ab + c\bar{c}/3)) + \cdots.$$

Now we restrict ourselves to the case where the system (3) is real. The solution of the system with the initial condition that $x = x_0$, $y = y_0$ when $t = 0$, can be given through the formula

$$x + iy = e^{it}z = e^{it}(T^*(A;t,0)z)_{z=z_0}$$

provided $z_0 = x_0 + iy_0$ is sufficiently close to 0. Let (x_1, y_1) be the point reached by the integral curve when $t = 2\pi$. Then

$$\begin{aligned} x_1 + iy_1 &= (T^*(A;2\pi,0)z)_{z=z_0} \\ &= z_0 + 2\pi z_0^2\bar{z}_0(k - 2i(2b\bar{b} - ab + c\bar{c}/3)) + \cdots. \end{aligned}$$

If $r_0^2 = x_0^2 + y_0^2$, then

$$\begin{aligned} x_1^2 + y_1^2 &= z_0\bar{z}_0 + 4\pi(\operatorname{Re} k - 2\operatorname{Im}(ab))(z_0\bar{z}_0)^2 + \cdots \\ &= r_0^2 + Kr_0^4 + \cdots. \end{aligned}$$

Hence we conclude that the integral curve of the autonomous system (3), after a time lapse of 2π, will carry every point in a sufficiently small neighborhood of (0, 0) in the real (x, y)-plane farther away or closer to (0, 0) according as $K > 0$ or $K < 0$.

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