FIBERINGS OF SPHERES AND H-SPACES WHICH ARE RATIONAL HOMOLOGY SPHERES

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Communicated by Alex Rosenberg, February 2, 1962

There are various fiber bundles known whose total spaces are spheres. Examples are the Hopf maps $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$ and and fiberings over projective spaces. It is an open question whether these are all the fiberings of spheres with connected fibers.

Spanier and Whitehead [**6**] showed that in such a fibering the fiber must be an H-space. The results of Borel [**2**; **3**] showed that the fiber is a rational homology sphere.

THEOREM 1. *Let $p: S^n \to B$ be a fiber bundle map onto a polyhedron B, with fiber F a connected polyhedron. Then F is the homotopy type of S^1, S^3 or S^7.*

The proof depends on studying the torsion of $H^*(F)$, in particular the 2-torsion.

THEOREM 2. *Let X be an H-space, connected, with $H_i(X)$ finitely generated for all i, zero for large i. Suppose in addition that X is a rational homology sphere, i.e., $H_*(X; Q) \cong H_*(S^n; Q)$ for some n. If $H^1(X; Z_2) = 0$, then*

$$H^*(X; Z_2) = \Lambda(x) \otimes Z_2[Sq^1 x]/(Sq^1 x)^{2^q}, \qquad 0 \leqq q < \infty,$$

$\dim x$ is odd, >1.

If $H^1(X; Z_2) \neq 0$, then $H^*(X; Z_2) = Z_2[x]/x^{2^q}$, $1 \leqq q < \infty$, $\dim x = 1$.

Since F satisfies the hypothesis of Theorem 2, we may apply it to this situation. A spectral sequence argument using Theorem 2, similar to Borel's argument [**3**, p. 165], yields the result that $H^*(F; Z_2) = \Lambda(x)$, i.e., F is a mod 2 homology sphere. Namely, Theorem 2 shows that $H^*(F; Z_2)$ has a simple system of transgressive generators, and employing a theorem of Borel [**3**, Proposition 16.1], we get that $H^*(B; Z_2)$ is a polynomial ring on the "transgressions" of the generators, in dimensions $<n$. But now, analyzing the structure of the spectral sequence, we cannot arrive at $E_\infty \cong H^*(S^n; Z_2)$, unless $H^*(F; Z_2)$ has only one generator; i.e., $H^*(F; Z_2) = \Lambda(x)$.

Adams [**1**] has shown that a mod 2 homology sphere which is an H-space is a 1, 3, or 7 dimensional mod 2 homology sphere, hence a rational homology 1, 3, or 7 sphere. It is then easy to show that an H-space which is a rational homology 1, 3, or 7 sphere has no odd torsion. Hence F is an integral homology sphere, and it follows that F is the homotopy type of a 1, 3, or 7 sphere.

Theorem 2 may also be applied to studying the cohomology of the "projective plane" of an H-space X, if X is a rational homology sphere. By studying the Steenrod squares in this space one can show that if $Sq^1x \neq 0$ then dim $x = 1$, from which one can deduce the following:

THEOREM 3. *Let X be an H-space, connected, with $H_i(X)$ finitely generated for all i, zero for large i, and suppose X is a rational homology sphere, i.e., $H_*(X; Q) \cong H_*(S^n; Q)$, for some n. Then X is the singular homotopy type of one of the following: S^1, S^3, S^7, P^3, or P^7, (P^i = real projective space of dimension i).*

The proofs of these theorems will be found in [**4**]. In addition, the proof of Theorem 2 relies heavily on some general results on differential Hopf algebras from [**5**].

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