tion is assumed with respect to ϕ. The space would not be an Orlicz space, but an extension of the L_p space for $p<1$. For the latter L_p spaces, it is known that the isometries are as described above.

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ON THE RECURRENCE OF SUMS OF RANDOM VARIABLES

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We give a very short proof of the recurrence theorem of Chung and Fuchs [1] in one and two dimensions. This new elementary proof does not detract from the old one which uses a systematic method based on the characteristic function and yields a satisfactory general criterion. But the present method, besides its brevity, also throws light on the combinatorial structure of the problem.

Let N denote the set of positive integers, M that of positive real numbers. Let $\{X_n, n\in \mathrm{N}\}$ be a sequence of independent, identically distributed real-valued random vectors, and let $S_n=\sum_{\nu=1}^{n} X_\nu$. The value x is possible iff for every $\epsilon>0$ there exists an n such that $P\{|S_n-x|<\epsilon\}>0$; it is recurrent iff for every $\epsilon>0$, $P\{|S_n-x|<\epsilon$ for infinitely many $n\}=1$. It is shown in [1] that every possible value is recurrent if and only if for some $m\in \mathrm{M}$ we have

$$(1)\qquad \sum_{n=1}^{\infty} P\{|S_n| < m\} = \infty.$$

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Note: In two dimensions $|x| = \max(|x_1|, |x_2|)$ if $x = (x_1, x_2)$. We are here concerned with obtaining sufficient conditions for the validity of (1). We state our results in two analogous propositions which correspond to one and two dimensions respectively.

PROPOSITION 1. *Let* $\{u_n(m): n \in \mathrm{N}, m \in \mathrm{M}\}$ *satisfy the following conditions*:

(i) *for each* n, $u_n(m)$ *is nonnegative, nondecreasing in* m *and* $\lim_{m\to\infty} u_n(m) = 1$;

(ii) *there exists a* $c > 0$ *such that for every positive integer* m,

$$\sum_{n=1}^{\infty} u_n(m) \leqq cm \sum_{n=1}^{\infty} u_n(1)$$

(*if the left member is infinite, the inequality is taken to mean that the right member must also be infinite*);

(iii) *for each* $\epsilon > 0$,

$$\lim_{n\to\infty} u_n(\epsilon n) = 1.$$

Then we have

$$\sum_{n=1}^{\infty} u_n(1) = \infty. \tag{2}$$

PROOF. Let $b \in \mathrm{N}$. We have by (i) and (ii), for integral m,

$$\sum_{n=1}^{\infty} u_n(1) \geqq \frac{1}{cm} \sum_{n=1}^{\infty} u_n(m) \geqq \frac{1}{cm} \sum_{n=1}^{bm} u_n(m) \geqq \frac{1}{cm} \sum_{n=1}^{bm} u_n\left(\frac{n}{b}\right).$$

Hence we have by (iii)

$$\sum_{n=1}^{\infty} u_n(1) \geqq \liminf_{m\to\infty} \frac{1}{cm} \sum_{n=1}^{bm} 1 = \frac{b}{c},$$

from which (1) follows since b is arbitrary, q.e.d.

REMARK. It is easy to see from the above proof that condition (iii) can be weakened, for example, to the following:

(iii*) there exists a $\delta > 0$ such that for every $\epsilon > 0$

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} u_\nu(\epsilon\nu) \geqq \delta.$$

PROPOSITION 2. *Let* $\{u_n(m): n \in \mathrm{N}, m \in \mathrm{M}\}$ *satisfy the condition* (i) *and* (ii$_2$) *there exists a* $c>0$ *such that for every positive integer* m

$$\sum_{n=1}^{\infty} u_n(m) \leqq cm^2 \sum_{n=1}^{\infty} u_n(1);$$

(iii$_2$) *there exist* $a>0$ *and* $d>0$ *such that*

$$u_n(m) \geqq \frac{dm}{n^{1/2}} \qquad \text{for } am^2 \leqq n.$$

Then we have (2) *again.*

PROOF. We have for $a<a'$,

$$\sum_{n=1}^{\infty} u_n(1) \geqq \frac{1}{cm^2} \sum_{am^2 \leqq n \leqq a'm^2} u_n(m) \geqq \frac{dm}{cm^2} \sum_{am^2 \leqq n \leqq a'm^2} \frac{1}{n^{1/2}}$$
$$\geqq \frac{d}{c}\left((a')^{1/2} - a^{1/2}\right)$$

for all sufficiently large m. Since a' is arbitrary, (2) follows, q.e.d.

APPLICATIONS. Take

$$u_n(m) = P\{\,|S_n| < m\}.$$

Condition (i) is trivially satisfied. Condition (ii) with $c=2$ or condition (ii$_2$) with $c=4$ is satisfied according as the X_n's are one-dimensional or two-dimensional. This known observation in renewal theory follows at once from the interpretation of $\sum_{n=1}^{\infty} u_n(m)$ as the expected number of entrances into the interval $(-m, m)$ by the random sequence $\{S_n, n \in \mathrm{N}\}$. Condition (iii) is the usual normalized form of the weak law of large numbers if $E(X_n)=0$, while condition (iii$_2$) is implied by the normalized form of the central limit theorem if $E(X_n)=0$ and $E(|X_n|^2)<\infty$. Note however that here we may use the validity of these classical limit theorems as our conditions.

Let us point out that in Theorem 4a of [1] the conditions are precisely those for the validity of the weak law of large numbers in the form (iii); in Theorem 5 there, the conditions of zero mean and finite variance do imply the central limit theorem in the required form.

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