ON THE COHOMOLOGY OF TWO-STAGE POSTNIKOV SYSTEMS

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1. **Introduction.** The purpose of this paper is to compute the cohomology of certain spaces with two nonvanishing homotopy groups. Let $P(\pi, n; \tau, m, k)(n<m)$ denote the space with homotopy groups π and τ in dimensions n and m, all other homotopy groups equal to zero, and (first) k-invariant equal to $k\in H^{m+1}(K(\pi, n), \tau)$. Let ϵ_i be the basic class in $H^i(K(\tau, i), \tau)$. We shall then compute the mod 2 cohomology of $P_{n,h}=P(Z_2, n, Z_2, 2^hn-1, \epsilon_n^{2^h})$.

Extending the methods of this paper, further computations can be carried out. This will be done in a subsequent paper.

2. **The Steenrod construction.** In this section we are working in the category of *css*-complexes. In the (non-normalized) chain complex $C_*(K)$ of a *css*-complex K we can define a filtration. Let namely σ_q denote a q-simplex in K. We can then in a unique way write σ_q in the form

$$\sigma_q = s_{i_1}s_{i_2}\cdots s_{i_{q-p}}\sigma_p, \qquad 0 \leqq i_{q-p} < \cdots < i_1 < q,$$

where σ_p is a nondegenerate p-simplex in K and s_i denotes a degeneracy operator in K. The generator $\sigma_q\in C_q(K)$ is then said to be of filtration p

$$\sigma_q \in F_pC_*(K).$$

This defines a filtration in $C_*(K)$.

Let π be a permutation group on the n letters $(0, 1, \cdots, n-1)$ and let V be an arbitrary π-free resolution of the integers. Let V be filtered by dimension. Let $V\otimes C_*$ and $C_*^{(n)}$ (the n-fold tensor product of C_*) be filtered by the usual tensor product filtration. Let π operate trivially in C_*, diagonally in $V\otimes C_*$, and by permutation of the factors in $C_*^{(n)}$. We then have the

THEOREM. *There exists a natural π-equivariant filtration and augmentation preserving transformation*

$$(1)\qquad \phi': V \otimes C_* \to C_*^{(n)}.$$

If Φ' is another such transformation then ϕ' and Φ' are homotopic by a natural π-equivariant homotopy of degree $\leqq 1$ (i.e. $H(v\otimes\eta)\in F_{p+i+1}$ if $\dim v=i$ *and $\eta\in F_p$).*

Let C denote the normalized cochain functor. Let $f: E \to B$ be an arbitrary *css*-mapping. Then f induces filtrations in $C_*(E)$ and $C(E)(C_*(E)$ is filtered by inverse images of skeletons in B. The filtration in $C(E)$ is (essentially) the dual of this filtration). The mapping (1) gives rise to a mapping

$$\phi: V \otimes_\pi C(E)^{(n)} \to C(E) \tag{2}$$

natural with respect to mappings

$$\begin{array}{ccc} E & \xrightarrow{g} & E_1 \\ f\downarrow & & \downarrow f_1 \\ B & \xrightarrow{\bar{g}} & B_1 \end{array}$$

with $\bar{g}f = f_1 g$. It is easy to see that ϕ has the property

$$\dim v = i,\ u_j \in F^{p_j} \Rightarrow \phi(v \otimes u_1 \otimes \cdots \otimes u_n) \in F^p \tag{3}$$

for $p \leqq \text{l.i.g.}(\max(1/n \sum_j p_j,\ \sum_j p_j - i))$, where l.i.g.$(\alpha)$ denotes the least integer greater than or equal to α. Defining the filtration in $V \otimes_\pi C^{(n)}$ according to (3) we have that ϕ preserves filtration.

3. **Operations in spectral sequences.** In the following we shall be working over the ground field Z_2 instead of the integers as above. Let us choose a mapping ϕ as in (2) and keep it fixed in the following.

Let $f: E \to B$ be a mapping of *css*-complexes. Let $x \in F^pC^{p+q} = F^pC^{p+q}(E)$. Then we define

$$sq^i x = \phi(e_{p+q-i} \otimes x^2 + e_{p+q-i+1} \otimes x dx). \tag{4}$$

Then

$$d sq^i x = sq^i dx. \tag{5}$$

Using standard notation we see that if $x \in Z_r^p$ then

$$\begin{aligned} sq^i x &\in Z_r^p, && \text{for } 0 \leqq i \leqq q - r + 1, \\ sq^i x &\in Z_{2r-1+i-q}^p, && \text{for } q - r + 1 \leqq i \leqq q, \\ sq^i x &\in Z_{2r-1}^{p+i-q}, && \text{for } q \leqq i \leqq p + q. \end{aligned} \tag{6}$$

If x represents a class $\bar{u}$ in $E_r^{p,q}$ then we can examine when the class of $sq^i x$ is independent of the choice of representative of $\bar{u}$. When this is the case we can define an operation in the spectral sequence. We get for $0 \leqq i \leqq q - r + i$,

$$Sq^i \bar{u} = \{sq^i x\} \in E_r^{p,q+i};$$

for $q-r+1 \leqq i \leqq q$,

$$Sq^i \bar{u} = \{sq^i x\} \in E_{r+j}^{p,q+i}, \qquad \text{for any } j,\ 0 \leqq j \leqq i-q+r-1,$$

for $q \leqq i \leqq p+q$,

$$Sq^i \bar{u} = \{sq^i x\} \in E_{r+j}^{p+i-q,2q}, \quad \text{for any } j,\ \min(i-q, r-2) \leqq j \leqq r-1.$$

These operations are natural, additive, and they commute with the differentials in the spectral sequence. Further we shall mention that if $\bar{u} \in E_r^{p,q}$, $d_r \bar{u} = 0$, and $\bar{u}$ determines $\{\bar{u}\} \in E_{r+1}^{p,q}$ then

$$\{Sq^i \bar{u}\} = Sq^i\{\bar{u}\} \in \begin{cases} E_{r+1}^{p,q+i} & \text{for } 0 \leqq i \leqq q, \\ E_{r+1+\min(i-q,r-1)}^{p+i-q,2q} & \text{for } q \leqq i \leqq p+q. \end{cases}$$

Let us suppose that $E_2^{*,0} \otimes E_2^{0,*} \to E_2^{*,*}$ is an isomorphism then in E_2 we have (denoting the homomorphism $a \to a^2$ by ζ)

$$Sq^i = 1 \otimes Sq^i: E_2^{p,q} \to E_2^{p,q+i}, \qquad \text{for } 0 \leqq i \leqq q,$$
$$Sq^i = Sq^{i-q} \otimes \zeta: E_2^{p,q} \to E_2^{p+i-q,2q}, \qquad \text{for } q \leqq i \leqq p+q.$$

If F is the fibre (relative to some base point in B) of the mapping $f: E \to B$, then we can consider cohomology operations in F, E, and B. Since we can use the mapping ϕ (2) to define these cohomology operations, they are in an obvious way related to the spectral operations considered here.

Operations in spectral sequences have also been constructed by S. Araki [1] and R. Vazquez [3]. The operations constructed in this paper coincide with or are related to the operations constructed in these papers.

4. **Some lemmas.** The following lemmas are crucial in the computation of $H^*(P_{n,h})$.

REMARK. Let $f: E \to B$ be a map of *css*-complexes and let $\{E_r, d_r\}$ be the corresponding spectral sequence. Let $\alpha \in E_n^{0,n-1}$, $\beta \in E_n^{n,0}$, and $\gamma \in E_n^{0,2(n-1)}$ $(n \geqq 2)$ with $d_n \alpha = \beta$, $d_n \gamma = \alpha\beta$. Let $E_j^{2n-j,j-1} = 0$, $j = 2, 3, \cdots, n-1$. Then there exist cochain representatives u, v, and x of α, β, and γ respectively with the property

$$dx = uv + a$$

with $a \in F^{2n-1}C^{2n-1}$ (we shall say that a is in the base. In general we shall say that any cochain belonging to $\sum_j F^j C^j$ is in the base.)

LEMMA. *Let $\alpha \in E_n^{0,n-1}$, $\beta \in E_n^{n,0}$, and $\gamma \in E_n^{0,2(n-1)}$ be elements in the spectral sequence $\{E_r, d_r\}$ associated with a css-map $f: E \to B$. Let u, v, and x be cochains representing α, β, and γ respectively with the properties*

$du=v$, $dx=uv+a$, where a is in the base. Then

$$\tau^{(2k+1)} = Sq^{2k+1}\gamma + \sum_{\sigma=0}^{k} Sq^{\sigma}\alpha Sq^{2k+1-\sigma}\alpha, \qquad 0 \leqq k < n-1,$$

is transgressive, while

$$\tau^{(2k)} = Sq^{2k}\gamma + \sum_{\sigma=0}^{k-1} Sq^{\sigma}\alpha Sq^{2k-\sigma}\alpha, \qquad 0 < k \leqq n-1,$$

persists to E_{n+k} and has

$$d_{n+k}\{\tau^{(2k)}\} = \{Sq^k\alpha\cdot Sq^k\beta\}.$$

Furthermore there are cochains u_1, v_1, and x_1 representing $Sq^k\alpha$, $Sq^k\beta$, and $\tau^{(2k)}$ respectively such that

$$du_1 = v_1 \quad \textit{and} \quad dx_1 = u_1v_1 + a_1,$$

where a_1 is in the base. (The existence of u_1, v_1, x_1, and a_1 with this property clearly implies (2).)

Also

$$\gamma\cdot d_n(\gamma) = \gamma\alpha\beta \in E_n^{n,3(n-1)}$$

is transgressive (i.e., persists till E_{3n-2}).

LEMMA. *Let $\alpha\in E_n^{0,n-1}$, $\beta\in E_n^{n,0}$, and $\gamma\in E_n^{0,2^hn-2}$ ($n\geqq 2$, $h\geqq 2$) be elements in the spectral sequence $\{E_r, d_r\}$ associated with a css-map $f\colon E\to B$. Let u, v, and x be cochains representing α, β, and γ respectively with the properties $du=v$, $dx=uv^{2^h}+a$ where a is in the base. Then*

$$Sq^k\gamma, \qquad k \leqq 2^hn-2,$$

is transgressive if n is not divisible by 2^h. If $k=s\cdot 2^h$, then

$$Sq^k\gamma = Sq^{s\cdot 2^h}\gamma$$

persists to $E_{(2^h-1)(n+s)}$ and has

$$d_{(2^h-1)(n+s)}\{Sq^{s\cdot 2^h}\gamma\} = \{Sq^s\alpha\cdot(Sq^s\beta)^{2^h-1}\}.$$

Furthermore there are cochains u_1, v_1, and x_1 representing $Sq^s\alpha$, $Sq^s\beta$, and $Sq^{s\cdot 2^h}\gamma$ respectively such that

$$du_1 = v_1, \qquad dx_1 = u_1v_1^{2^h-1} + a_1,$$

with a_1 in the base. Also

$$\gamma\cdot d_{(2^h-1)n}(\gamma) = \gamma\alpha\beta^{2^h-1} \in E_{(2^h-1)n}$$

is transgressive (i.e. persists till $E_{(2^h+1)n-2}$).

5. **Computations.** Using the Moore comparison theorem for spectral sequences and the above mentioned results $H^*(P_{n,h})$ can be derived. We shall use the usual notation and properties of sequences $I=(a_1, a_2, \cdots, a_r)$ of non-negative integers (see e.g. Serre [2]). In particular we use the notation

$$L(d, h) = (2^{h-1}d, 2^{h-2}d, \cdots, d).$$

THEOREM. *Let $P_n=P(Z_2, n; Z_2, 2n-1, \epsilon_n^2)$. For each admissible sequence $J, e(J)\leqq 2(n-1)$, containing odd components and each admissible sequence N, $e(N)<n-1$, there are classes $\beta(J)$ and $\gamma(2N)$ in $H^*(P_n)$ of dimensions $2n-1+\deg J$ and $2(2n-1+2\deg N)$ respectively, satisfying*

$$\beta(J) = Sq^{\bar{J}}(\beta((2j+1)J_1))$$

whenever $J=\bar{J}(2j+1)J_1$ with all components of J_1 even. Let α be the nonzero class in $H^*(P_n)$, then

$$H^*(P_n) = Z_2[\{\beta(J)\}] \otimes \Lambda(\{Sq^I\alpha\}) \otimes Z_2[\{Sq^{L(4(n-1+\deg N),h)}\gamma(2N)\}],$$

where $h=0, 1, \cdots$ and where J, I, and N run through all admissible sequences satisfying $e(J)\leqq 2(n-1)$, $e(I)\leqq n-1$, and $e(N)<n-1$; further it is required that J contains odd components.

THEOREM. *Let $P_{n,h}=P(Z_2, n; Z_2, 2^hn-1, \epsilon_n^{2^h})$ $(n\geqq 2, h\geqq 2)$. For each admissible sequence J, $e(J)\leqq 2^hn-2$, $J\not\equiv 0 \pmod{2^h}$, and for each admissible sequence I, $e(I)\leqq n-1$, there are classes $\beta(J)$ and $\gamma(I)$ in $H^*(P_{n,h})$ of dimensions $2^hn-1+\deg J$ and $2^{h+1}(n+\deg I)-2$ respectively, satisfying*

$$\beta(J) = Sq^{\bar{J}}(\beta((j)J_1))$$

whenever $J=\bar{J}(j)J_1$ with $j\not\equiv 0 \pmod{2^h}$ and $J_1\equiv 0 \pmod{2^h}$. Let α be the nonzero class in $H^n(P_{n,h})$ then

$$H^*(P_{n,h}) = Z_2[\{\beta(J)\}] \otimes Z_2[\{Sq^I\alpha\}, 2^h] \otimes Z_2[\{\gamma(I)\}],$$

where $Z_2[\{x_i\}, 2^h]$ denotes the truncated polynomial algebra of height 2^h in the generators $\{x_i\}$ $(x_i^{2^h}=0)$, and where J and I run through all admissible sequences satisfying $e(J)\leqq 2^hn-2$, $J\not\equiv 0 \pmod{2^h}$, and $e(I)\leqq n-1$.

It is of some interest to get the complete action of the Steenrod algebra A^* in $H^*(P_{n,h})$. At the present, however, we only have scattered information about this action of A^*.

A detailed account will appear elsewhere.

REFERENCES

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