ANALYTIC CONTINUATION OF THE PRINCIPAL SERIES

BY R. A. KUNZE AND E. M. STEIN[1]

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The purpose of this note is to announce results obtained in the analytic continuation of the (nondegenerate) "principal series" of representations of the $n\times n$ complex unimodular group. This study has as its starting point a similar one for the 2×2 real unimodular group previously carried out by us in [4].

We let G be the $n\times n$ complex unimodular group and C its diagonal subgroup consisting of elements $c=(c_1, c_2, \cdots, c_n)$. A continuous character λ of C is given by

$$\lambda(c) = \left(\frac{c_1}{|c_1|}\right)^{m_1} \cdots \left(\frac{c_n}{|c_n|}\right)^{m_n} |c_1|^{s_1} \cdots |c_n|^{s_n}$$

where the sequences of integers $m_1, m_2, \cdots, m_n$ and complex numbers $s_1, s_2, \cdots, s_n$ are uniquely determined by setting $0\leqq m_1+m_2+\cdots+m_n<n$ and $s_1+s_2+\cdots+s_n=0$. Notice that λ is unitary, i.e., has values in the circle group, if $\operatorname{Re}(s_j)=0$, $j=1, 2, \cdots, n$. Gelfand and Neumark have shown how to construct for each unitary λ an irreducible unitary representation $a\rightarrow T(a, \lambda)$ of the group G [2]. To describe these representations (i.e., the principal series) we follow the method but not the notation of [2].

Let V be the subgroup of G of elements having ones on the main diagonal and zeros above the main diagonal. Then G acts on V in a natural way; we denote the action of $a\in G$ on $v\in V$ by $v\tilde{a}$ (the transformations $v\rightarrow v\tilde{a}$ are linear fractional transformations when $n=2$ and generalizations thereof in higher dimensions). The operators of the representation $T(\cdot, \lambda)$ are given by

$$T(a, \lambda)f(v) = m(v, a; \lambda)f(v\tilde{a})$$

where $m(v, a; \lambda)$ is an appropriate multiplier, and the underlying Hilbert space is $L_2(V)$.

In order to state our results we introduce a tube $\mathfrak{T}$ lying in the complex hyperplane $s_1+s_2+\cdots+s_n=0$. The base B of $\mathfrak{T}$ is the smallest convex set which contains the points $(\sigma, -\sigma, 0, 0, \cdots, 0)$, $-1<\sigma<1$ and is invariant under all permutations of coordinates. A

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point $(s_1, s_2, \cdots, s_n)$, $s_j=\sigma_j+it_j$ belongs to $\mathfrak{Z}$ if and only if $s_1+s_2+\cdots+s_n=0$ and $(\sigma_1, \sigma_2, \cdots, \sigma_n)\in B$.

THEOREM 1. *For each character* $\lambda=\lambda(m_1, m_2, \cdots, m_n; s_1, s_2, \cdots, s_n)$ *for which* $(s_1, s_2, \cdots, s_n)\in\mathfrak{Z}$ *there exists a representation* $a\rightarrow R(a, \lambda)$ *of* G *on* $L_2(V)$. *The family of all such representations has the following properties*:

(1) *For each fixed* λ, $a\rightarrow R(a, \lambda)$ *is a continuous uniformly bounded representation.*

(2) *For fixed* a, $R(a, \lambda)$ *as a function of* $(s_1, s_2, \cdots, s_n)$ *is analytic in the tube* $\mathfrak{Z}$.

(3) *When* λ *is unitary*, $R(\cdot, \lambda)$ *is unitarily equivalent to* $T(\cdot, \lambda)$, *the corresponding member of the principal series.*

(4) $R(\cdot, \lambda)=R(\cdot, p\lambda)$ *for every "permutation"* p *of the character* λ.

(5) $R(\cdot, \lambda)'=R(\cdot, \lambda')$ *where* $R(a, \lambda)'=R(a^{-1}, \lambda)^*$ *and* $\lambda'(c)=\bar{\lambda}(c^{-1})$.

In their work on the principal series Gelfand and Neumark obtained a trace formula for certain operators associated with the representations $a\rightarrow T(a, \lambda)$, [2, p. 73]. The formula involves a function (the character of the representation) which we denote by $\psi(a, \lambda)$. It is initially defined for unitary characters λ but extends in an obvious way by analyticity to nonunitary characters. We prove the following result.

THEOREM 2. *Let* λ *be as in Theorem* 1 *and* f *be the convolution of two bounded functions on* G *with compact support. Then the operator*

$$R(f, \lambda) = \int_G f(a)R(a, \lambda)da$$

is of trace class, and

$$\operatorname{tr} R(f, \lambda) = \int_G f(a)\psi(a, \lambda)da.$$

REMARKS. (i) It is known that $T(\cdot, \lambda)$ and $T(\cdot, p\lambda)$ are unitarily equivalent for every permutation p [2, p. 118]. In the present context this unitary equivalence becomes an identity by (4) of Theorem 1.

(ii) It follows almost immediately from (4) and (5) that $R(\cdot, \lambda)$ is unitary if there exists a permutation p such that $\lambda'=p\lambda$, where λ' is the contragredient of λ defined by $\lambda'(c)=\bar{\lambda}(c^{-1})$. If $\lambda'=p\lambda$ and p equals the identity we are dealing with the principal series. However, when p is not the identity and $\lambda'=p\lambda$, it then follows from Theorem 2 that $R(\cdot, \lambda)$ is unitarily equivalent to a member of the complementary series [3].

(iii) By means of Theorem 2 we also show that if $\lambda' \neq p\lambda$ for all p, then the representation $R(\cdot, \lambda)$ is not equivalent to a unitary one, although it is uniformly bounded. The existence of a group $(SL(2, R))$ and uniformly bounded representations of it not equivalent to unitary ones was first proved by Ehrenpreis and Mautner [1].

We shall briefly describe the ideas behind the proofs of Theorems 1 and 2. We begin by limiting our attention to unitary characters λ. A basic fact we prove can be stated as follows:

LEMMA. *Let G_0 be the subgroup of G consisting of the matrices a such that $a_{jn}=0$, $1 \leqq j \leqq n-1$. For any character λ, call the integer r, uniquely determined by $0 \leqq r < n$, $r = m_1 + m_2 + \cdots + m_n$, the residue of the character λ. Then if λ_1 and λ_2 have the same residue the restrictions of $T(\cdot, \lambda_1)$ and $T(\cdot, \lambda_2)$ to G_0 are unitarily equivalent.*

Thus when we restrict to G_0, there are only n (essentially) different representations among the principal series. This is remarkable in view of the known fact that the members of the principal series are all irreducible on G_0 [2, p. 22]. Hence there is a unitary operator $W(\lambda)$ so that if we set

$$R(a, \lambda) = W(\lambda)T(a, \lambda)W^{-1}(\lambda)$$

then $R(a, \lambda)$ for fixed $a \in G_0$ depends only on the residue of λ. We call the representations $R(\cdot, \lambda)$ the *normalized principal series.* It is these that can be continued analytically (i.e., to nonunitary λ's) as in Theorem 1, while the $T(\cdot, \lambda)$ cannot.

The actual construction of the operators $W(\lambda)$ is too complicated to describe here but is intimately connected with the construction of the intertwining operators $A(p, \lambda)$ between $T(\cdot, \lambda)$ and $T(\cdot, p\lambda)$. In fact, it follows from part (4) of Theorem 1 that up to a constant multiple

$$A(p, \lambda) = W^{-1}(p\lambda)W(\lambda).$$

Moreover, we show that the operator $W(\lambda)$ can be written as a product of $n(n-1)/2$ operators of the type $A(p, \lambda)$.

The work described above has many points of contact with the analysis of the special case of the 2×2 (real) group which we carried out previously in [4]. However, there is an essential difference. This is due to the fact that the study of the representations in question is closely related to the Fourier analysis on $L_2(V)$. When $n=2$, V is a commutative group; but this is not so when $n>2$, and therein lies the major obstacle to the proof of Theorem 1.

REFERENCES

1. L. Ehrenpreis and F. I. Mautner, *Uniformly bounded representations of groups*, Proc. Nat. Acad. Sci. U.S.A. vol. 41 (1955) pp. 231–233.

2. I. M. Gelfand and M. A. Neumark, *Unitäre Darstellungen der klassischen Gruppen*, Berlin, Akademie-Verlag, 1957.

3. Harish-Chandra, *Representations of semi-simple Lie groups*. III, Trans. Amer. Math. Soc. vol. 76 (1954) pp. 234–253.

4. R. A. Kunze and E. M. Stein, *Uniformly bounded representations and harmonic analysis of the* 2×2 *real unimodular group*, Amer. J. Math. vol. 82 (1960) pp. 1–62.

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THE UNIVERSITY OF CHICAGO