OBSTRUCTIONS TO THE EXISTENCE OF ALMOST COMPLEX STRUCTURES

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1. **Definitions and notation.** Let M be an orientable, differentiable manifold of dimension $2n$ and let $\xi = (E_\xi, M, R^{2n}, \pi)$ denote the tangent bundle of M; we assume the structural group of ξ has been reduced from the full linear group to the special orthogonal group $SO(2n)$. By definition, M admits an almost complex structure if and only if the associated fibre bundle $\eta = (E, M, \Gamma_n, p)$ admits a cross section;[1] here Γ_n denotes the homogeneous space $SO(2n)/U(n)$. In this paper, we will study the obstructions to a cross section for any fibre bundle $\theta = (E, B, \Gamma_n, p)$ with structural group $SO(2n)$ and base space B a CW-complex. If $s: B^q \to E$ is a cross section of θ over the q-skeleton of the base space B, then the obstruction to extending s over the $(q+1)$-skeleton is denoted by

$$c^{q+1}(s) \in H^{q+1}(B, \pi_q(\Gamma_n)).$$

Since θ is a bundle with structural group $SO(2n)$, the following characteristic classes are defined:

(a) Integral Stiefel-Whitney classes,

$$W_i(\theta) \in H^i(B, Z), \qquad 3 \leqq i \leqq 2n-1, \qquad i \text{ odd}.$$

(Recall that $2 \cdot W_i(\theta) = 0$.)

(b) Euler-Poincaré class, $W_{2n}(\theta) \in H^{2n}(B, Z)$.

(c) Pontrjagin classes $p_i(\theta) \in H^{4i}(B, Z)$, $0 \leqq i \leqq n$.

In an analogous manner, if ξ is a fibre bundle with base space B and structural group $U(n)$, the Chern classes of ξ will be denoted by $c_i(\xi) \in H^{2i}(B, Z)$, $0 \leqq i \leqq n$.

2. **Statement of results.** The homotopy group $\pi_q(\Gamma_n)$ is called *stable* if $q < 2n-1$; it is well known that the stable homotopy groups $\pi_q(\Gamma_n)$ for fixed q and variable n are all isomorphic; see Gray [4, p. 432]. The stable homotopy groups of Γ_n have been determined by Bott [2]; he showed that in the stable range,

[1] Standard references on the subject of almost complex structures are Ehresmann's lecture at the 1950 International Congress of Mathematicians [3] and the last section of Steenrod's book [10].

The author would like to take this opportunity to acknowledge that his proof of the two theorems announced in Abstract 60T-24, Notices Amer. Math. Soc. vol. 7 (1960) p. 1001, contains an apparently irreparable gap. Whether or not these two theorems are correct is not known.

$$\pi_q(\Gamma_n) = Z \qquad \text{for } q \equiv 2 \bmod 4,$$
$$(1) \qquad \pi_q(\Gamma_n) = Z_2 \qquad \text{for } q \equiv 0 \text{ or } -1 \bmod 8,$$
$$\pi_q(\Gamma_n) = 0 \qquad \text{for all other values of } q.$$

LEMMA 1. *The first nonstable homotopy group* $\pi_{2n-1}(\Gamma_n)(n>0)$ *is as follows*:

$$\pi_{2n-1}(\Gamma_n) = \begin{cases} Z + Z_2 & \text{for } n \equiv 0 \bmod 4, \\ Z_{(n-1)!} & \text{for } n \equiv 1 \bmod 4, \\ Z & \text{for } n \equiv 2 \bmod 4, \\ Z_{(n-1)!/2} & \text{for } n \equiv 3 \bmod 4. \end{cases}$$

This result for $n \equiv 0 \bmod 4$ is due to Bruno Harris. The results for the other three cases are easier; they are proved by considering the homotopy sequences of some well-known fibre bundles and using the results listed in a paper of Kervaire [9].

Recall that the vanishing of the integral Stiefel-Whitney classes $W_{2q+1}(\theta)$ is a well-known *necessary* condition for the existence of a cross section of the fibre bundle $\theta = (E, B, \Gamma_n, p)$ (see Steenrod [10, p. 212]). On the other hand in the stable range the obstruction to a cross section in dimension $4k+3$ will be an integral cohomology class in view of (1). It is natural to conjecture that there should be some relation between this obstruction and the Stiefel-Whitney class $W_{4k+3}(\theta)$.

THEOREM I. *Let* $s: B^q \to E$ *be a cross section of the bundle* θ *over the* q*-skeleton, where* $q = 4k+2$ *and* $q < 2n-1$. *Then*

$$W_{q+1}(\theta) = \begin{cases} (2k)!c^{q+1}(s) & \text{for } k \text{ even}, \\ (1/2)(2k)!c^{q+1}(s) & \text{for } k \text{ odd}. \end{cases}$$

REMARK 1. For $q=2$, this theorem asserts that $W_3(\theta) = c^3(s)$, which is of course well known. For $q=6$, the result becomes $W_7(\theta) = c^7(s)$, a result announced by Ehresmann without proof in 1950 [3].

REMARK 2. In case $H^{q+1}(B, Z)$ has no p-torsion for any prime $p \leq 2k$, then the condition $W_{q+1}(\theta) = 0$ implies that $c^{q+1}(s) = 0$.

REMARK 3. This theorem bears a slight similarity to formula (ii) of Lemma (1, 1) of Kervaire [8]. The proof here is more difficult because Γ_n is not a topological group and θ is not a principal bundle.

REMARK 4. This theorem implies divisibility conditions on the integral Stiefel-Whitney classes. For example, if $\theta = (E, B, p)$ is a bundle with group $SO(2n)$, $n \geq 6$, such that $H^8(B, Z_2) = H^9(B, Z_2) = 0$ and $W_3(\theta) = W_7(\theta) = 0$, then $W_{11}(\theta)$ is divisible by 24.

As motivation for the next theorem, recall that if the bundle θ admits a cross section s, then the structural group can be reduced from $SO(2n)$ to the subgroup $U(n)$. Let ξ denote the $U(n)$ bundle thus defined (ξ depends on the cross section s) and $c_i(\xi)$, $1 \leqq i \leqq n$, its Chern classes. Then the following relations must hold between the Pontrjagin classes $p_i(\theta)$ and the Chern classes $c_i(\xi)$:

$$(-1)^k p_k(\theta) = \sum_{i+j=2k} (-1)^i c_i(\xi) c_j(\xi), \qquad 0 \leqq k \leqq n \tag{2}$$

(see Hirzebruch, [6, Satz 4.5.1, p. 68]). In addition, the top Chern class and the Euler-Poincaré class are equal:

$$W_{2n}(\theta) = c_n(\xi). \tag{3}$$

Now assume that n is even, $n=2k$, and that $s: B^{2n-1} \to E$ is a cross section of θ over the $2n-1$ skeleton. The obstruction $c^{2n}(s) \in H^{2n}(B, \pi_{2n-1}(\Gamma_n))$ is an integral class if $n \equiv 2 \bmod 4$, while $c^{2n}(s) = c_0^{2n}(s) + c_2^{2n}(s)$ if $n \equiv 0 \bmod 4$, where c_0^{2n} is an integral class and c_2^{2n} is a mod 2 cohomology class.

THEOREM II. *For $n=2k$, k odd,*

$$\sum_{i+j=2k} (-1)^i c_i(\xi) c_j(\xi) - (-1)^k p_k(\theta) = 4 \cdot c^{2n}(s)$$

while for $n=2k$, k even, this same formula holds true with $c^{2n}(s)$ replaced by its integral component, $c_0^{2n}(s)$. In this formula, $c_0(\xi), \cdots, c_{n-1}(\xi)$ are the Chern classes of the $U(n)$ bundle ξ induced over B^{2n-1} by s, while $c_n(\xi) = W_{2n}(\theta)$.

Theorems I and II give information about the obstructions to a cross section of θ in all cases of importance where the coefficient group is infinite cyclic. Further information is needed in case the coefficient group is Z_2. The first such case is the following: Assume $s: B^7 \to E$ is a cross section of $\theta = (E, B, \Gamma_n, p)$ over the 7-skeleton and $n > 4$. Then $c^8(s)$ is a mod 2 cohomology class. The existence of s implies that $W_3(\theta)$, $W_5(\theta)$, and $W_7(\theta)$ vanish, and that the fibre Γ_n is totally nonhomologous to zero in dimensions $\leqq 8$ with any coefficients.[2] ($H^*(\Gamma_n, Z)$ is torsion free.) In dimensions $\leqq 8$, $H^*(\Gamma_n, Z)$ is a polynomial ring[2] on generators $x \in H^2(\Gamma_n, Z)$ and $y \in H^6(\Gamma_n, Z)$. It follows that there exist elements $u \in H^2(E, Z)$ and $v \in H^6(E, Z)$ such that

$$i^*(u) = x, \qquad i^*(v) = y, \tag{4}$$

where $i: \Gamma_n \to E$ is the inclusion map.

[2] These assertions follow easily from the facts about the cohomology of Γ_n stated in the next section.

LEMMA 2. *Given the cross section* s, *it is possible to choose* u *and* v *so that* (4) *is satisfied and* $s^*(u)=s^*(v)=0$. *Conversely, given the cohomology classes* u *and* v *satisfying* (4), *there exists a cross section* $s: B^7 \to E$ *such that* $s^*(u)=s^*(v)=0$.

Next, it may be shown that $Sq^2y=x^4$ (mod 2). Since the fibre is totally nonhomologous to 0 in dimensions $\leqq 8$, there exist unique mod 2 cohomology classes b_2, b_2', b_4, b_6, b_8 on B such that

$$(5) \qquad \begin{aligned} Sq^2v = u^4 &+ p^*(b_8) + p^*(b_6)\cdot u + p^*(b_4)\cdot u^2 \\ &+ p^*(b_2)\cdot u^3 + p^*(b_2')\cdot v \pmod 2 \end{aligned}$$

(the subscripts denote the degree).

THEOREM III. *If* u *and* v *are chosen to satisfy* (4) *and* $s^*(u)=s^*(v)=0$, *and* $b_8 \in H^8(B, Z_2)$ *is chosen to satisfy* (5), *then*

$$c^8(s) = b_8.$$

This theorem essentially asserts that determination of $c^8(s)$ requires the computation of $Sq^2: H^6(E, Z) \to H^8(E, Z_2)$, where $H^*(E)$ is considered as a module over $H^*(B)$. This computation is at present a very difficult problem.

An easy computation using Lemma 2 shows that if s_0, $s_1: B^7 \to E$ are cross sections, then there exist cohomology classes $d_2 \in H^2(B, Z)$ and $d_6 \in H^6(B, Z)$ such that

$$(6) \qquad c^8(s_0) - c^8(s_1) = Sq^2d_6 + (d_2)^4 \pmod 2.$$

Moreover, given s_0, d_2, and d_6, there exists a cross section s_1 such that this equation holds true.

COROLLARY. *If* $\theta=(E, B, \Gamma_n, p)$ *is a bundle with structural group* $SO(2n)$, $n>4$ *such that* $W_3(\theta)=W_7(\theta)=0$ *and*

$$H^8(B, Z_2) = Sq^2H^6(B, Z) + Sq^4Sq^2H^2(B, Z)$$

then θ *admits a cross section over the* 8-*skeleton of* B.

3. Some remarks on the proof of these theorems. We use the method of R. Hermann [5] to study the obstructions to cross sections of the bundle $\theta=(E, B, \Gamma_n, p)$. This method utilizes a Moore-Postnikov decomposition of the fibre space θ, which in turn requires some knowledge of a Postnikov decomposition of the fibre Γ_n. In order to use a Postnikov decomposition of Γ_n it is necessary to study the cohomology of Γ_n. The following are the relevant facts:

(a) Γ_n is torsion free; additively, its integral cohomology groups are isomorphic to those of the following product of even dimensional spheres:

$$S^2 \times S^4 \times S^6 \times \cdots \times S^{2n-2}$$

(see Borel, [1, p. 203]).

(b) The integral cohomology ring $H^*(\Gamma_n, Z)$ has a simple system of generators (in the sense of Borel, [1, p. 141]) $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$, with α_i of degree $2i$. These generators satisfy the following relations, which completely determine the structure of the integral cohomology ring:

$$\begin{aligned} &\alpha_1^2 - \alpha_2 = 0, \\ &\alpha_2^2 - 2\alpha_1\alpha_3 + \alpha_4 = 0, \\ &\alpha_3^2 - 2\alpha_2\alpha_4 + 2\alpha_1\alpha_5 - \alpha_6 = 0, \\ &\cdot \\ &\cdot \\ &\cdot \\ &\alpha_{n-2}^2 - 2\alpha_{n-3}\alpha_{n-1} = 0, \\ &\alpha_{n-1}^2 = 0. \end{aligned}$$

(c) In any fibre bundle $\theta = (E, B, \Gamma_n, p)$ with group $SO(2n)$, the generators $\alpha_1, \cdots, \alpha_{n-1}$ listed above are transgressive; the transgression of the generator α_i is the integral Stiefel-Whitney class $W_{2i+1}(\theta)$ (modulo the ideal generated by $W_3, \cdots, W_{2i-1}$).

(d) In the fibre space $p: B_{U(n)} \to B_{SO(2n)}$ determined by the inclusion $U(n) \subset SO(2n)$, the homomorphism $i^*: H^*(B_{U(n)}, Z) \to H^*(\Gamma_n, Z)$, where $i: \Gamma_n \to B_{U(n)}$ is the inclusion map, satisfies

$$\begin{aligned} i^*(c_j) &= 2\alpha_j, \\ i^*(c_n) &= 0. \end{aligned} \qquad 1 \leqq j \leqq n-1,$$

Here $c_j \in H^{2j}(B_{U(n)}, Z)$ denotes the universal Chern class.

(e) From (c), one can determine the Steenrod squares in $H^*(\Gamma_n, Z_2)$. It is necessary to reduce modulo 2 and use the known formulas of W. T. Wu for the squares of the Stiefel-Whitney classes.

(f) It follows from (d) that $i^*: H^*(B_{U(n)}, Z_p) \to H^*(\Gamma_n, Z_p)$ is a homomorphism onto for any odd prime p; hence the formulas of Serre and Borel for the Steenrod reduced powers in $H^*(B_{U(n)}, Z_p)$ determine those in $H^*(\Gamma_n, Z_p)$.

From these facts, one can determine enough information about the Postnikov invariants of Γ_n to prove Theorems I, II and III by Hermann's method; full details will be published elsewhere.[3]

[3] These results may perhaps be regarded as a first small step in the program mentioned by Hirzebruch in the middle of p. 127 of his 1958 lecture [7].

Note that all our results extend to analogous theorems about the almost contact manifolds of Gray [4]; for example, an orientable 7-dimensional manifold admits an almost contact structure if and only if $W_3=0$ (since $W_7=0$ automatically).

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