AN EXAMPLE IN SUMMABILITY

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A matrix A is called *conservative* if Ax is convergent (its limit is called $\lim_A x$) whenever x is a convergent sequence, *regular* if $\lim_A x = \lim x$ for such x, *coregular* if conservative and $\chi(A) \equiv \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} - \sum_{k=1}^{\infty} a_k \neq 0$ (here $a_k = \lim_{n\to\infty} a_{nk}$), and *conull* if $\chi(A) = 0$. The terms coregular and conull were introduced in [2].

A regular matrix is coregular, as is any matrix equipotent with a regular one. However, there exist coregular matrices not equipotent with any regular matrix. The example, due to Zeller, is given in [3]. We present here an example of a quite different nature.

(An open problem in the field is that of characterizing FK spaces which have a right to be called coregular. That $\{1\}$ be separated from the linear closure of $\{\delta^n\}$ is necessary but not sufficient.)

Restricting ourselves, for convenience, to triangles ($a_{nn} \neq 0$, $a_{nk} = 0$ for $k > n$), let $c_A = \{x: Ax \text{ is convergent}\}$. Then c_A is isomorphic with c, the space of convergent sequences, under $A: c_A \to c$. Thus c_A becomes a Banach space. If $c_A = c_B = F$ say, the norms on F due to A, B are equivalent since $A = DB$ with $c_D = c$ and $\|x\|_A = \|Ax\| \leqq \|D\| \, \|Bx\| = \|D\| \, \|x\|_B$.

If the functional lim is continuous on $c \subset c_A$ we extend it by the Hahn-Banach theorem to be defined on all of c_A. By a construction of Mazur [1, Theorem 2, p. 45], we obtain a matrix B with $\lim_B = \lim$ on c, and $c_B = c_A$. (See [2] for proof that lim satisfies Mazur's condition.)

Clearly B is regular.

Conversely if such regular B exists it follows that lim is continuous since $\lim = \lim_B$.

Thus, for our example, it is sufficient to construct a coregular matrix A such that lim is not continuous on c_A.

Let Y be the matrix such that $Yx = \{x_{n-1} + x_n\}$. Then $(1/2)Y$ is a regular triangle. Let B be the matrix whose nth row is $\{t_1, t_2, \cdots, t_{n-1}, 0, 0, \cdots\}$ where $\{t_n\}$ is a suitably chosen sequence with $\sum |t_n| < \infty$. Then B is in the radical of the Banach algebra Δ of conservative triangular matrices. (See [4].) Note that Y has no inverse in this algebra. Finally, let $A = B + Y$. Then A is coregular. The norm associated with c_A is

$$\|x\| = \sup_n \left| \sum_{k=1}^{n-1} t_k x_k + x_{n-1} + x_n \right|.$$

We shall choose $t_n = (-1)^n/n^2$. Now to construct x with $|\lim x|$ large and $\|x\|$ not large we shall, given any integer m, supply x with $\lim x = -1/2 \log m$, $\|x\| < M$ where M is an absolute constant which could easily be determined.

Namely, let

$$\begin{aligned} x_n &= n(-1)^n - 1/2 \ \log n \qquad \text{for } 1 \leqq n \leqq m, \\ &= (2m - n)(-1)^n - 1/2 \ \log m \qquad \text{for } m < n \leqq 2m, \\ &= -1/2 \ \log m \qquad \text{for } n > m. \end{aligned}$$

If the matrix Y had been chosen so that $Yx = \{sx_{n-1} + tx_n\}$ then for $t > s \geqq 0$ we have that $Y^{-1} \in \Delta$ hence $A^{-1} \in \Delta$ since B is in the radical. Thus $c_A = c$, lim is continuous and $\|x\| < 1$ implies $|\lim x|$ is less than some constant, depending on s, t and $\{t_n\}$. This result also follows from Lemma 4.2 of [4]. There should be situations between these extremes in which $c_A \neq c$, yet a regular matrix coincident with A exists. This may very well occur if all $t_n > 0$, and $s = t = 1$, but we are unable to say at present.

Added in proof. Research Problem 3 (Bull. Amer. Math. Soc. vol. 67 (1961) p. 355) by Albert Wilansky, which inspired this article, has also been solved by Lawrence Shepp.

References

1. S. Mazur, *Eine Anwendung der Theorie der Operationen bei der Untersuchung der Toeplitzchen Limitierungsverfahren*, Studia Math. vol. 2 (1930) pp. 40–50.

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3. ———, *Summability: the inset, replaceable matrices, the basis in summability space*, Duke Math. J. vol. 19 (1952) pp. 647–660.

4. A. Wilansky and K. Zeller, *Banach algebra and summability*, Illinois J. Math. vol. 2 (1958) pp. 378–385.

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