COHOMOLOGY OF MAXIMAL IDEAL SPACES

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Let A be a commutative Banach algebra with unit, and let M be the maximal ideal space of A. We say that A is generated by $x_1, \cdots, x_n$ if the polynomials $p(x_1, \cdots, x_n)$ form a dense subalgebra of A. Let $H^j(M, C)$ denote the jth Čech cohomology group of M with complex coefficients.

THEOREM. *If A is generated by n elements, then $H^j(M,C)=0$ for $j \geqq n$.*

Proof. If $x_1, \cdots, x_n$ generate A, then the map of M into C^n given by $h \rightarrow (h(x_1), \cdots, h(x_n))$ is a homeomorphism of M onto a compact set K. It is known (see, e.g., [1]) that K is polynomially convex, i.e., if V is any open set containing K, there exists an analytic polyhedron U defined by polynomials, such that $K \subset U \subset V$. Each such polyhedron U is a domain of holomorphy (Stein manifold) and a Runge domain. For any n-dimensional Stein manifold U, it is known that $H^j(U, C)=0$ for $j>n$. (See [2] for a proof.) For any Runge domain U in C^n, Serre has shown [3] that $H^n(U, C)=0$. The proof is completed by observing the following nonstandard but elementary continuity property of Čech cohomology:

FACT. Let X be a compact subset of a metric space, G an abelian group, j a non-negative integer. If for every open set $V \supset K$, there exists an open U with $K \subset U \subset V$ and $H^j(U, G)=0$, then $H^j(K, G)=0$.

COROLLARY. *Let M be an n-dimensional compact orientable manifold. Let $C(M)$ denote the ring of all continuous complex-valued functions on M, normed by the sup norm. Then $C(M)$ requires at least $n+1$ generators.*

REMARKS. 1. For $n=1$, the condition of the theorem is both necessary and sufficient; a compact subset K of the plane is polynomially convex if and only if K has connected complement, which is equivalent to $H^1(K, C)=0$.

2. It is of course trivial that at least $n+1$ real-valued functions are required to generate $C(M)$ when M is a compact n-dimensional manifold, but it should be observed that in general, a compact space X need not require as many complex functions to generate $C(X)$ as it does real functions. Example: If X is a compact connected plane set

with no interior and connected complement, $C(X)$ is generated by the single function z (Mergelyan's theorem); but $C(X)$ is generated by a single real function if and only if X is a Jordan arc.

3. The author is unaware of any other proof of the corollary even for the case M the two-sphere.

References

1. R. Arens and A. P. Calderón, *Analytic functions of several Banach algebra elements*, Ann. of Math. vol. 62 (1955) pp. 204–216.
2. Séminaires de H. Cartan, 1951–1952.
3. J.-P. Serre, *Une propriété topologique des domaines de Runge*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 133–134.

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