# FLOWS ON SOME THREE DIMENSIONAL HOMOGENEOUS SPACES

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1. Flows on surfaces of constant negative curvature have been investigated for some time. The geodesic flow [**3**] and the horocycle flow [**2**] have known minimal and ergodic properties. These flows may be looked at as flows induced on a three dimensional homogeneous space by a one parameter subgroup of a Lie group [**4**]. This idea has been carried further in [**1**; **5**] where one parameter flows on general nilmanifolds are studied.

The manifolds considered here are all compact manifolds of the form $G/D$ where $G$ is a noncompact connected, simply connected three dimensional Lie group and $D$ a discrete uniform subgroup. If $\phi\colon T\to G$ is a one parameter subgroup of $G$, then the one parameter flow defined by $t(gD)=\phi(t)gD$, is an action of the reals on $G/D$. The classification as to which of these flows are minimal and which are ergodic is now complete. In this note we outline this classification; complete proofs will be presented elsewhere.

There are only three cases to consider: simple, nilpotent, and solvable but not nilpotent.

2. $G$ **simple.** If $G$ is simple and noncompact then its Lie algebra $\mathfrak{G}$ is isomorphic to the Lie algebra of the two by two real matrices with trace zero. Each one parameter subgroup of $G$ is of the form $\phi(t)=\exp \overline{X}t$, where $\overline{X}\in\mathfrak{G}$. Let $G(2)$ be the group of all $2\times 2$ real matrices of determinant one. $G$ is the universal covering group of $G(2)$ and we let $\eta$ be the covering homomorphism $\eta\colon G\to G(2)$.

THEOREM 1. *If $D$ is a discrete uniform subgroup of $G$ then the mapping $\psi\colon G/D\to G(2)/\eta(D)$ given by $\psi(gD)=\eta(g)\eta(D)$ is a finite covering and $\eta(D)$ is discrete.*

THEOREM 2. *Let $G$ be the connected, simply connected, noncompact, three dimensional, simple Lie group; and let $D$ be a discrete uniform subgroup of $G$; and let $\phi(t)=\exp \overline{X}t$. The following statements hold:*

(1) *If $\overline{X}$ has real nonzero eigenvalues the one parameter flow induced*

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*by $\phi(t)$ on $G/D$ has infinitely many closed orbits, and is thus not minimal. This flow is ergodic.*

(2) *If $\overline{X}$ has only zero eigenvalues then the one parameter flow induced by $\phi(t)$ on $G/D$ is minimal and ergodic.*

(3) *If $\overline{X}$ has nonzero complex eigenvalues then the one parameter flow induced by $\phi(t)$ on $G/D$ is equivalent to an action on $G/D$ by the circle group and is thus neither minimal nor ergodic.*

Theorem 1 is used to reduce the proof of Theorem 2 to the geodesic flow [**3**; **4**] or the horocycle flow [**2**].

3. $G$ **nilpotent.** The complete classification of flows on nilmanifolds has been worked out in [**1**; **5**] and, if $[G, G]$ is the commutator of $G$, it reads as follows:

THEOREM 3. *If $G$ is a connected, simply connected nilpotent Lie group; and $D$ a discrete uniform subgroup; and $\phi(t)$ a one parameter subgroup; then the flow induced by $\phi(t)$ on $G/D$ is always distal. Furthermore, it is ergodic (minimal) if and only if the flow induced on the torus $G/D[G, G]$ by $\phi(t)$ is ergodic (minimal).*

4. $G$ **solvable.** We let $G_1$ be the set of matrices of the form

$$\begin{pmatrix} \cos 2\pi z & \sin 2\pi z & 0 & x \\ -\sin 2\pi z & \cos 2\pi z & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z$ are real numbers. We let $G_2$ be the set of matrices of the form

$$\begin{pmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z$ are real numbers, and $k$ is a fixed nonzero real number such that $e^{-k}+e^{k}$ is an integer.

THEOREM 4. *If $G$ is a connected, simply connected, three dimensional, solvable Lie group and $D$ is a discrete uniform subgroup, then one of the following is true.*

(1) *$G$ is nilpotent.*

(2) *$G$ is isomorphic to $G_1$ and $D$ is generated by*

$$\begin{pmatrix}1&0&0&u_1\\0&1&0&u_2\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad\begin{pmatrix}1&0&0&v_1\\0&1&0&v_2\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad\begin{pmatrix}\cos 2\pi n/p&\sin 2\pi n/p&0&0\\-\sin 2\pi n/p&\cos 2\pi n/p&0&0\\0&0&1&n\\0&0&0&1\end{pmatrix}$$

*where $n$ is a fixed integer, $p$ is either 2, 3, 4, or 6, and*

$$\begin{vmatrix}u_1&u_2\\v_1&v_2\end{vmatrix}\neq 0.$$

(3) *$G$ is isomorphic to $G_1$ and $D$ is generated by*

$$\begin{pmatrix}1&0&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad\begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad\begin{pmatrix}1&0&0&u_1\\0&1&0&u_2\\0&0&1&n\\0&0&0&1\end{pmatrix},$$

*where $n$ is a fixed integer and $u_1$ and $u_2$ are fixed real numbers.*

(4) *$G$ is isomorphic to $G_2$ and $D$ is generated by*

$$\begin{pmatrix}1&0&0&u_1\\0&1&0&u_2\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad\begin{pmatrix}1&0&0&v_1\\0&1&0&v_2\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad\begin{pmatrix}e^{kn}&0&0&0\\0&e^{-kn}&0&0\\0&0&1&n\\0&0&0&1\end{pmatrix}$$

where $n$ is a fixed integer and

$$\begin{vmatrix}u_1&u_2\\v_1&v_2\end{vmatrix}\neq 0.$$

If, for the sake of brevity, we write the matrices of $G_1$ and $G_2$ as columns

$$\begin{pmatrix}x\\y\\z\end{pmatrix}$$

then the one parameter subgroups of $G_1$ are in one of the following forms.

(1) $$\begin{pmatrix}at\\bt\\0\end{pmatrix},\qquad a \text{ and } b \text{ are real numbers.}$$

(2) $$\begin{pmatrix}a\sin 2\pi ct+b[\cos 2\pi ct-1]\\b\sin 2\pi ct-a[\cos 2\pi ct-1]\\ct\end{pmatrix},$$

$a$, $b$, $c$, real numbers and $c \neq 0$. The one parameter subgroups of $G_2$ have the following forms.

$$\begin{pmatrix} at \\ bt \\ 0 \end{pmatrix}, \qquad a \text{ and } b \text{ real numbers.} \tag{1}$$

$$\begin{pmatrix} a(e^{kct} - 1) \\ b(e^{-kct} - 1) \\ ct \end{pmatrix}, \qquad a, b \text{ and } c \text{ real numbers} \tag{2}$$

and $c \neq 0$.

In either $G_1$ or $G_2$ we refer to these as one parameter groups of the first and second type respectively.

THEOREM 5. *If $G$ is a connected, simply connected, three dimensional, non-nilpotent solvable Lie group, $D$ a discrete uniform subgroup, and $\phi: T \to G$ a one parameter subgroup, then one of the following is true.*

(1) *If $G$ is isomorphic to $G_1$, $D$ is as in Theorem* 4 *number* (2), *and $\phi$ is of the first type, then the flow is neither ergodic nor minimal. If $\phi$ is of the second type, then the flow is equivalent to the action of a circle group and is thus neither ergodic nor minimal.*

(2) *If $G$ is isomorphic to $G_2$, $D$ as in Theorem* 4 *number* (4), *and $\phi$ is of the first type, then the flow is neither ergodic nor minimal. If $\phi$ is of the second type then the flow is ergodic and has a closed orbit and is thus not minimal.*

(3) *If $G$ is isomorphic to $G_1$, $D$ as in Theorem* 4 *number* (3), *and $\phi$ is of the first type, then the flow is neither ergodic nor minimal. If $\phi$ is of the second type, then the flow is equivalent to a straight line flow on the three dimensional torus and thus has the same minimal and ergodic properties.*

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