RESEARCH PROBLEMS

4. Richard Bellman: *Theory of numbers*

The question of the solubility in rational integers of the congruence $x^2+ax+b\equiv 0(p)$ can be decided by examining a polynomial congruence $a^2-4b\equiv y^2(p)$. Similarly, one can study the solubility of $x^3+ax+b\equiv 0(p)$ and $x^4+ax+b\equiv 0(p)$.

It is conjectured that the solubility in rational integers of $x^5+ax+b\equiv 0(p)$ cannot be discussed in terms of any finite system of polynomial congruences involving the coefficients a and b, where the number of congruences and the degrees are independent of the prime p. (Received May 1, 1961.)

5. Richard Bellman: *Control processes*

Consider the problem of minimizing the quadratic functional

$$J(y) = \int_0^T [(x, Ax) + 2(x, By) + (y, Cy))]dt,$$

over all vector functions y related to x by means of the linear differential equation $dx/dt=Ax+y$, $x(0)=c$, and subject to the component constraints $|y_i|\leqq m_i$, $i=1, 2, \cdots, N$. Can one obtain an explicit analytic solution? (Received May 1, 1961.)

6. Herbert S. Wilf: *Reciprocal bases for the integers*

It is well known that every integer is the sum of reciprocals of distinct integers. Let us call a sequence S: $\{n_1, n_2, n_3, \cdots\}$ of distinct integers an R-basis if every integer is the sum of reciprocals of finitely many integers of S. It is clearly necessary that

$$\sum n_j^{-1} = \infty$$

though this is not sufficient, as can be seen by considering the primes. Yet it is not necessary to use all the integers, since $a, 2a, 3a, \cdots$ will obviously do, for any a.

Are the odd numbers an R-basis? Is every arithmetic progression an R-basis? Does an R-basis necessarily have a positive density? lower density? If S contains all integers and $f(n)$ is the least number required to represent n, what, in some average sense, is the growth of $f(n)$? (Received June 12, 1961.)

COMMUTATIVE BANACH ALGEBRA PROBLEMS

The following list of problems was assembled in a small conference on commutative Banach algebras held at Dartmouth College in August, 1960, sponsored jointly by the National Science Foundation and the Air Force Office of Scientific Research. (Received May 6, 1961.)

7. Let X be compact metric, let A be a non-self-adjoint closed subalgebra with unit of $C(X)$, and suppose X is the maximal ideal space of A. The function $f \in A$ "peaks" at $x \in X$ if $|f(y)| < |f(x)|$ for every $y \in X$, $y \neq x$. Must there exist some $x \in X$ such that no $f \in A$ peaks at x?

8. Take X and A as in Problem 7. Let K be a compact subset of X, and let A_K be the uniform closure on K of the restriction of A to K. Suppose $g \in A_K$ peaks (see Problem 7) at an interior point x of K. Does it follow that some $f \in A$ also peaks at x?

9. Take X and A as in Problem 7. Let $x, y \rightarrow f(x, y)$ be a continuous function on the product space of X with itself, and suppose that for each fixed $x_0 \in X$ the functions $x \rightarrow f(x, x_0)$ and $x \rightarrow f(x_0, x)$ both belong to A. Does it follow that $x \rightarrow f(x, x)$ also belongs to A?

10. Take X and A as in Problem 7. Must there exist a real Radon measure on X orthogonal to all $f \in A$?

11. Let W be the unit circle $|w| = 1$ in the complex plane, and let A be a non-self-adjoint closed subalgebra with unit of $C(W)$. Suppose no real Radon measure on W is orthogonal to all $f \in A$, i.e., the real parts of members of A are uniformly dense in $C(W)$. Is A then isomorphic to the analytic functions on the open disk $|z| < 1$ with continuous boundary values?

12. Let Z be the closed unit disk $|z| \leqq 1$ and let A be a closed subalgebra of $C(Z)$ such that for each proper closed subset K of Z we have $C(K) = A_K$, defined as in Problem 8. Does it follow that $A = C(Z)$? Note that the answer to this question is negative if instead of the disk Z we have the circle W.

13. Let Z be as in Problem 12 and let A be a closed subalgebra of $C(Z)$ having Z as maximal ideal space. Does the Silov boundary of Z with respect to A necessarily intersect the ordinary boundary $|z| = 1$? The "Silov boundary" is the smallest closed subset L of Z such that $\sup_{z \in L} |f(z)| = \sup_{z \in Z} |f(z)|$ for all $f \in A$.

14. A compact subset K of complex n-space is "polynomial-con-

vex" if for every z outside K there is a polynomial f with $|f(z)| > \sup_{w \in K} |f(w)|$. If L is the Silov boundary of a polynomial-convex K with respect to the algebra of polynomials, is then $K-L$ locally the union of analytic varieties of positive dimension? (This problem has since been solved in the negative by Gabriel Stolzenberg.)

15. Let K be a polynomial-convex compact subset of complex n-space (see Problem 14). Call the function f "locally analytic on K" if every $z \in K$ is the center of a ball U on which is defined an analytic function F that agrees with f on $U \cap K$. Can such an f be approximated uniformly on K by polynomials?

16. Given in complex n-space an open set U which is a domain of holomorphy and is homeomorphic to the ball, can every function f analytic on U be approximated uniformly on compact subsets of U by rational functions which are analytic on U?

17. Let W be the circle group, and let A be the convolution algebra $L^1(W)$, regarded as a pointwise algebra on the integers Z. Any self-adjoint closed subalgebra B determines a partition of the integers into finite cells Z_k, the sets of constancy of the members of B. The partition $\{Z_k\}$ itself determines a closed subalgebra B', consisting of all $g \in A$ constant on each Z_k. Clearly $B' \supseteq B$, but is $B' > B$ possible?

18. Let W be the circle group. For each $p > 1$, what are all the automorphisms of the convolution algebra $L^p(W)$? There are many that do not correspond to automorphisms of $L^1(W)$.

19. Let W be the unit circle. Let A be the algebra of all functions on W with absolutely convergent Fourier series. The following properties of A are known:

(i) Only analytic functions operate: If g is a function defined in some open set V in the complex plane which is not real-analytic, then there is an $f \in A$ with range in V such that the composite $g \circ f$ does not belong to A.

(ii) Spectral synthesis fails: There is an $f \in A$ that cannot be approximated in the A-norm by functions $g \in A$ which vanish on neighborhoods of the set Z of zeroes of f.

(iii) Stone-Weierstrass fails: There is a closed self-adjoint proper subalgebra of A which separates the points of W.

(iv) The only automorphisms of A are induced by rotations and reflections of W.

Consider now any self-adjoint Banach function algebra B on W which is invariant under rotations and reflections, which contains A

as a dense subalgebra, but which is smaller than the algebra of all continuous functions on W. Must some of the properties (i), (ii), (iii), (iv) hold also for the algebra B? To prove (iv) it may be necessary to assume more, for instance that the mapping $w \to w^2$ of W into W induces an endomorphism of B into B.

20. For any f and Z which exhibit the failure of spectral synthesis as in (ii) of Problem 19, there will exist a pseudomeasure λ on the circle (i.e., a Schwartz distribution with bounded Fourier coefficients) with support in Z and with $\int f\lambda = 1$. Can we always demand that the Fourier coefficients of λ tend to 0 at infinity? Or, at the other extreme, can Z be a set of uniqueness (i.e., a set that supports no pseudomeasure with Fourier coefficients 0 at infinity)? Malliavin's pseudomeasure μ does have Fourier coefficients 0 at infinity, and there are $g \in A$ with $\int g\mu = 1$ and $g \equiv 0$ on the support Z of μ. But one can ask another interesting question about Malliavin's g. Does at least one of them belong to a weak-star-closed ideal of functions in A vanishing identically on Z? Here we regard A as the dual of the space of sequences which vanish at infinity.

21. Let T be any group, let $t \to A_t$ be a bounded representation of T on a reflexive Banach space X, let $x \in X$, $y \in X'$. Then $t \to (A_t x, y)$ is weakly almost-periodic. Do all weakly almost-periodic functions on T arise in this way?

22. Let T be any group. Does the space of weakly almost-periodic functions on T have an invariant mean? (If T is commutative, the answer is yes. On noncommutative T, the space of uniformly almost-periodic functions always has an invariant mean, but the space of all bounded functions need not have one.)

23. Does an unbounded derivation on a non-semi-simple commutative Banach algebra A necessarily map the radical R into itself?

24. Does there exist a commutative Banach algebra R which is all radical but in which every element has a square root?

25. Under what conditions does the "Wedderburn principal theorem" ($A = R \oplus B$, with $B \approx A/R$) hold for a commutative Banach algebra A with radical R? Does it hold, for instance, when every n-fold product in R is zero, and A/R is a $C(X)$? (Perhaps one cannot demand that B be closed.)

26. Which commutative semi-simple Banach algebras A have nontrivial homomorphisms into radical Banach algebras R? In particular,

suppose A is the algebra $C_0(X)$ of all continuous functions vanishing at infinity on a locally compact space X, or the convolution algebra $L^1(G)$ of all integrable functions on non-compact locally compact abelian group G.

27. Let A be a commutative complex algebra with a "locally multiplicative" Fréchet topology. That is, A is an algebra, and is a complete locally convex space with a countable neighborhood basis $\{V_k\}$ at 0 such that $V_k^2 \subset V_k$. Is every multiplicative linear functional on A continuous?