VECTOR FIELDS ON SPHERES

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1. The problem is to determine the maximal number of the independent continuous fields of tangent vectors on the unit n-sphere S^n. The number will be denoted by $\lambda(n)$.

$\lambda(n)$ is the maximal number of k such that the boundary homomorphism $\Delta_{n,k}: \pi_n(S^n) \to \pi_{n-1}(O_{n,k})$ associated with the fibering $O_{n+1,k+1}/O_{n,k} = S^n$ is trivial, where $O_{n,k}$ denotes the Stiefel manifold of the orthogonal k-vectors (k-frames) in the real n-space R^n.

The fundamental conjecture for our problem is stated as follows.

CONJECTURE. Does $\lambda(n) = \lambda^*(n)$ for all $n > 0$?

Here, the conjectured values $\lambda^*(n)$ are defined as follows:

$$\lambda^*(n) = \lambda_r, \qquad \text{if } n \equiv 2^r - 1 \pmod{2^{r+1}},$$
$$\lambda_0 = 0, \qquad \lambda_1 = 1, \qquad \lambda_2 = 3, \qquad \lambda_3 = 7$$

and

$$\lambda_{r+4} = \lambda_r + 8.$$

It was known that the conjecture is true for the cases $r = 0, 1, 2, 3$ [4].

The obtained results on $\lambda(n)$ are the following.

THEOREM 1. (a) $\lambda^*(n) \leqq \lambda(n)$. (b) *If* $k = \lambda^*(n)$, *then the image of* $\Delta_{n,k}: \pi_n(S^n) \to \pi_{n-1}(O_{n,k})$ *coincides with the image of the composition* $i_* \circ J: \pi_k(SO(n-k-1)) \to \pi_{n-1}(S^{n-k-1}) \to \pi_{n-1}(O_{n,k})$ *of G. Whitehead's homomorphism* J *and the homomorphism* i_* *induced by the usual injection* $i: S^{n-k-1} \subset O_{n,k}$.

The first part (a) is provided by the recent work of Bott and Shapiro, *Clifford modules and vector fields on spheres* (mimeographed note), which states the existence of a continuous field of linear $\lambda^*(n)$-frames on S^n.

THEOREM 2. $\lambda^*(n) = \lambda(n)$ *if* $n \equiv 2^r - 1 \pmod{2^{r+1}}$ *for an integer* $r < 11$.

Then our problem is still open in question on the sphere S^{2047}.

THEOREM 3. $\lambda(2^i m - 1) \geqq \lambda(m-1) + 2^{i-1}$ *for* $i = 1, 2, 3, 4$.

COROLLARY. *If the above conjecture is not true for an* $n \equiv 2^r - 1 \pmod{2^{r+1}}$ *and* $r = 4s - 1$ (s: *positive integer*), *then the conjecture is not true for all* n *of* $r \geqq 4s - 1$.

2. The following lemma means that our problem is a stable one, that is, we may assume that for each r the integer n is sufficiently large.

LEMMA 1. *$\lambda^*(2^r-1) \leqq \lambda(2^r-1)$ if and only if $\lambda^*(n) \leqq \lambda(n)$ for an integer n of $n \equiv 2^r-1 \pmod{2^{r+1}}$.*

This lemma is proved by applying the theory [3] of James.

In the following, we always assume that the integers n are sufficiently large with respect to the other integers k and i such that the homology and homotopy considered are stable. Then we can replace $O_{n,k}$ by a cell complex

$$P_{n,k} = P^{n-1}/P^{n-k-1} = S^{n-k} \cup e^{n-k+1} \cup \cdots \cup e^{n-1}$$

which is obtained from real projective $(n-1)$-space P^{n-1} by shrinking its $(n-k-1)$-subspace P^{n-k-1} to a point, since the cellular decomposition of $O_{n,k}$ given in [7] shows that $P_{n,k}$ is a subcomplex of $O_{n,k}$ and the dimensionalities of $O_{n,k}-P_{n,k}$ are greater than $2n-2k$. The exact sequence for the fibering $O_{n+1,k+1}/O_{n,k}=S^n$ is replaced by the following exact sequence:

$$\cdots \to \pi_n(S^n) \xrightarrow{\Delta_{n,k}} \pi_{n-1}(P_{n,k}) \xrightarrow{i_*} \pi_{n-1}(P_{n+1,k+1}) \to \cdots.$$

Now our problem is transformed to a problem on the homotopy of $P_{n+1,k+1}$.

LEMMA 2. *$\lambda(n)$ is a maximal number of k such that the attaching map of the n-cell $e^n = P_{n+1,k+1}-P_{n,k}$ is inessential in $P_{n,k}$, namely, $\Delta_{n,k}(\iota_n)=0$ for the class $\iota_n \in \pi_n(S^n)$ of the identity of S^n.*

The following two lemmas are obtained by translating results of [3] in our words.

LEMMA 3. *If $k \leqq \lambda(m-1)$, then the m-fold iterated suspension $E^m P_{n,k+1}$ of $P_{n,k+1}$ and $P_{n+m,k+1}$ have the same homotopy type.*

In fact, a homotopy equivalence is given by the composition of the join: $E^m P_{n,k+1} = P_{n,k+1} * S^{m-1} \to P_{n,k+1} * P_{m,k+1}$ of the identity and a cross-section: $S^{m-1} \to P_{m,k+1}$ with the intrinsic join: $P_{n,k+1} * P_{m,k+1} \to P_{m+n,k+1}$.

LEMMA 4. *Let n be odd, $h \leqq k \leqq \lambda(n)$ and n be large ($n \geqq 2(k+h)$). Assume that $\Delta_{n,k+h}(\iota_n)$ is the image of $\alpha \in \pi_{n-1}(P_{n-k,h})$ under the homomorphism i_* induced by the injection $i: P_{n-k,h} \subset P_{n,k+h}$, then $\Delta_{2n+1,k+h}(\iota_{2n+1})$ is the image of $2E^{n+1}\alpha \in \pi_{2n}(P_{2n+1-k,h})$ under the homomorphism i'_* induced by the injection $i': P_{2n+1-k,h} \subset P_{2n+1,k+h}$.*

Briefly stated, $\Delta_{n,k+h}(\iota_n)$ is an obstruction to the inequality $k+h \leqq \lambda(n)$ and two times it is an obstruction to $k+h\leqq\lambda(2n+1)$, since E^{n+1} is an isomorphism.

3. Let $K_{n,k}$ be a simply connected finite cell complex having the same homology of $P_{n,k}$. As n is so large, then $K_{n,k}$ has the same homotopy type as a suspension of a complex. The set of all the homotopy classes of the mappings of $K_{n,k}$ in itself forms a group as in [1]. Let $\iota_{n,k}$ be the class of the identity of $K_{n,k}$.

LEMMA 5. *Let n be odd and large ($n>4k$). Then $2^t\iota_{n,2k}=0$ for $\lambda_{t-1} \geqq 2k-1$. For example, $4\iota_{n,2}=0$, $8\iota_{n,4}=0$ and $16\iota_{n,6}=16\iota_{n,8}=0$.*

This is proved by giving deformations in $K_{n,2k}$ and by applying the results on the stable groups $\pi_{n+k}(S^n)$ for $n\leqq 7$.

COROLLARY. *Let n be odd and large with respect to k and i. Then $2^t\pi_{n+i}(K_{n,2k})=0$ and $2^t\pi^{n+i}(K_{n,2k})=0$ for $\lambda_{t-1}\geqq 2k-1$, where π^{n+i} denotes the $(n+i)$th cohomotopy group.*

Now Theorem 3 is a consequence of this corollary and Lemma 4. As another application of this corollary, we have the following:

THEOREM 4. *Let n be odd and $\lambda_{t-1}\geqq 2k-1$. Let $\Omega^{2k}(S^{n+2k})$ be the $2k$-fold iterated loop-space of S^{n+2k}. Then $2^t\pi_i(\Omega^{2k}(S^{n+2k}), S^n)=0$ for $i\leqq 3n-2$. If $i\leqq 4n-3$, then $2^t\pi_i(\Omega^{2k}(S^{n+2k}), S^n)$ has no 2-torsion.*

By a similar method to J-homomorphism [6] of G. Whitehead, we have a homomorphism $J: \pi_{i-1}(E^{n-1}P_{n+2k,2k})\rightarrow\pi_i(\Omega^{2k}(S^{n+2k}), S^n)$, which is an isomorphism if $i\leqq 3n-2$ and an isomorphism of the 2-primary components if $i\leqq 4n-3$. Then the theorem is proved by the above corollary.

As a corollary of Theorem 4, we have similar statements for the kernel and cokernel of $E^{2k}: \pi_i(S^n)\rightarrow\pi_{i+2k}(S^{n+2k})$.

4. Denote by $J_k\subset\pi_{n+k}(S^n)$ the image of G. Whitehead's J-homomorphism $J: \pi_k(SO(n))\rightarrow\pi_{n+k}(S^n)$. Applying Bott's periodicity $\Omega^8 SO(\infty)=SO(\infty)$, the following lemma is proved.

LEMMA 6. *Let n be large. Let η be the generator of J_1, and let σ^h and ζ^h be generators of J_{8h-1} and J_{8h+3} respectively, $h=1, 2, \cdots$. Then J_{8h} and J_{8h+1} are generated by the compositions $\sigma^h\circ\eta$ and $\sigma^h\circ\eta\circ\eta$, respectively, and ζ^h and σ^{h+1} are represented by compositions $g\circ f: S^{n+8h+3} \rightarrow S^{n+8h-1}\cup e^{n+8h+3}\rightarrow S^n$ and $g'\circ f': S^{n+8h+7}\rightarrow S^{n+8h-1}\cup e^{n+8h+7}\rightarrow S^n$, respectively, where $g\,|\,S^{n+8h-1}$ and $g'\,|\,S^{n+8h-1}$ represent σ^h, f and f' induce* *homomorphisms of degree 2 of the homology groups and the cells* e^{n+8h+8} *and* e^{n+8h+7} *are attached to* S^{n+8h-1} *by essential mappings.*

The proof of Theorem 1 is done by induction on r. Assume that Theorem 1 is proved for an $r=4h-1$, $h\geqq 1$. In this case, $P_{n-8h-1,3}$ is of the same homotopy type as $P_{n-8h-2,2}\vee S^{n-8h-2}$ and $\Delta_{n,8h+2}(\iota_n)=i_*(\beta'+\sigma^h)$ for $\beta'\in\pi_{n-1}(P_{n-8h-2,2})$ and $\sigma^h\in\pi_{n-1}(S^{n-8h-2})$. In the complex $P_{n-8h-2,2}=S^{n-8h-4}\cup e^{n-8h-3}$ the cell e^{n-8h-3} is attached by degree 2. Then by shrinking S^{n-8h-4} to a point, the element β' goes to an element $\beta\in\pi_{n-1}(S^{n-8h-3})$ such that $2\beta=0$. Thus $\Delta_{n,8h+1}(\iota_n)=i_*(\beta+\sigma^h)$ for $2\beta=0$. Consider $2i_*(\beta+\sigma^h)=i_*(2\sigma^h)$. Since the cell $e^{n-8h-1}=P_{n-8h,3}-P_{n-8h-1,2}$ is attached to $P_{n-8h-1,2}=S^{n-8h-3}\vee S^{n-8h-2}$ by a mapping which represents η at S^{n-8h-3} and is of degree 2 at S^{n-8h-2}, it follows that $2\sigma^h\in\pi_{n-1}(S^{n-8h-2})$ and $\sigma^h\circ\eta\in\pi_{n-1}(S^{n-8h-3})$ go to the same element by the injections into $P_{n-8h,3}$. By Lemma 4, we have that $\Delta_{2n+1,8h+1}(\iota_{2n+1})=i_*E^{n+1}(\sigma^h\circ\eta)$ and this is the statement of Theorem 1 for $r=4h$.

Next step of the proof is done by showing that if $\Delta_{n,8h+2}(\iota_n)=i_*\alpha$ for an element $\alpha\in\pi_{n-1}(P_{n-8h-2,2})$ which goes to $\sigma^h\circ\eta$ by shrinking S^{n-8h-4}, then $\sigma^h\circ\eta\circ\eta\in\pi_{n-1}(S^{n-8h-4})$ goes to 2α by the injection of S^{n-8h-4} into $P_{n-8h-2,2}$.

The remaining two steps of the proof of Theorem 1 are too complicated to describe here, but it can be done by applying above lemmas.

5. The proofs of Theorem 2 are purely computations of the homotopy groups $\pi_{n-1}(P_{n,k})$ for $k=\lambda^*(n)+1$, by showing that $\Delta_{n,k}(\iota_n)\neq 0$. In the computations, the following results on the homotopy groups of spheres and several relations in them are used [5]. Let $(G_k;2)$ be the 2-primary component of the stable group $\pi_{n+k}(S^n)$, $n>k+1$; then we have the following table.

$k=$	$(G_k;2)=$
7	Z_{16}
8	Z_2+Z_2
9	$Z_2+Z_2+Z_2$
10	Z_2
11	Z_8
12	0
13	0
14	Z_2+Z_2
15	$Z_{32}+Z_2$
16	Z_2+Z_2
17	$Z_2+Z_2+Z_2+Z_2+Z_2$
18	Z_8+Z_2
19	Z_8+Z_2

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COMPACT KAEHLER MANIFOLDS WITH POSITIVE RICCI TENSOR

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The purpose of the present note is to announce the following:

THEOREM 1. *A compact Kaehler manifold with positive definite Ricci tensor is simply connected.*

We say that the first Chern class of a compact Kaehler manifold is positive definite if it can be represented by a real closed (1, 1)-form which is positive in the sense of Kodaira [2]. The first Chern class of a manifold satisfying the assumption in Theorem 1 is necessarily positive definite. Theorem 1 follows from the following two theorems.

THEOREM 2. *If the first Chern class of a compact Kaehler manifold M is positive definite, then the fundamental group of M has no proper subgroup of finite index.*

THEOREM OF MYERS. *The fundamental group of a compact Riemannian manifold with positive definite Ricci tensor is finite* [3].

Theorem 2 can be proved by Kodaira's Vanishing Theorem and by the Riemann-Roch Theorem of Hirzebruch. Let g_p be the dimension of the space of holomorphic p-forms on M. Then $\chi(M)=\sum_{p=0}^{n}(-1)^p g_p$, where $n=\dim_C M$, is called the arithmetic genus of M. If M is