ON THE EULER CHARACTERISTIC OF COMPACT COMPLETE LOCALLY AFFINE SPACES. II

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The main result of this paper may be stated very simply: *A compact complete locally affine space has Euler characteristic zero.* In [1] we showed that if the radical of the fundamental group is nontrivial then the Euler characteristic is zero. Hence all that remains is to show that the radical of the fundamental group is indeed nontrivial. To do this one may as well limit oneself to the study of those compact completely locally affine spaces where the holonomy group is discrete and isomorphic to the fundamental group. For it is known that in all other cases the radical is nonzero (see [3]).

Let $A(n) = \mathrm{GL}(n) \cdot R^n$ be the group of affine transformations of the n-dimensional affine plane V^n, where the dot denotes the semi-direct product and R^n is the n-dimensional vector space. Let Γ be a discrete subgroup of $A(n)$ such that V^n/Γ is a compact manifold. From [2], it is trivial that every compact complete locally affine space can be so realized. Hence we have the holonomy group of Γ, $h(\Gamma)$, isomorphic to $\Gamma R^n/R^n \subset \mathrm{GL}(n)$, is discrete and isomorphic to Γ. Now $\Gamma\backslash A(n)$ can be identified with the principal bundle of all frames over M. Further, $\Gamma\backslash A(n)$ is sheeted by the images of the cosets of V^n in $A(n)$. This sheeting, since $h(\Gamma)$ is discrete, determines a fiber bundle over $h(\Gamma)\backslash\mathrm{GL}(n)$ with fiber homeomorphic to V^n. We will call this fiber bundle B. Since the fiber is solid there exists a cross-section $\psi: h(\Gamma)\backslash\mathrm{GL}(n) \rightarrow B$. Further each fiber inherits the structure of an affine plane. In addition this affine structure is preserved by the mappings of the fiber onto itself induced by the group of the bundle. Since the cross-section ψ exists, it is easy to verify that the group of the bundle is $h(\Gamma)$ acting on the fiber.

It is possible using the holonomy covering space in [2] to define $h(\Gamma)$ acting on the bundle B. Further for $h(\gamma) \in h(\Gamma)$ $h(\gamma): B \rightarrow B$ is a bundle map which acts trivially on the base space. Hence $h \cdot \psi$ is another section over $h(\Gamma)\backslash\mathrm{GL}(n)$. Therefore the vector determined by $h \cdot \psi$ is invariant under $h(\Gamma)$ acting on the fiber. Hence, this shows that the images of the origin under Γ acting on V^n are left point-wise fixed by $h(\Gamma)$ acting on V^n. Since the images of the origin under Γ must span V^n for V^n/Γ to be compact, it follows that B is a trivial bundle. Using this fact one can show that Γ must be abelian.

Complete details will be presented elsewhere.

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A GENERAL CLEBSCH-GORDAN THEOREM

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Relative to a Cartan decomposition of a simple Lie algebra over the complex field and an ordering of the roots, let W be the Weyl group, ϕ half the sum of the positive roots, and $P(\beta)$ the number of partitions of β as a sum of positive roots. In a fairly complicated way, Kostant [2] has proved that the multiplicity of μ as a weight in the irreducible representation with highest weight λ is

$$m_\lambda(\mu) = \sum_{s \in W} \det s \; P(s(\phi + \lambda) - (\phi + \mu)). \tag{1}$$

Cartier [1] and the present author have noticed, independently, that Weyl's character formula and (1) are simple formal consequences of each other. (Incidentally, Cartier seems to be wrong in saying that Kostant's work thus provides another algebraic proof of Weyl's formula, since the latter is Kostant's starting point for the proof of (1).) In this note we deduce from (1) the following explicit formula for the multiplicity of an irreducible representation in the tensor product of two others. If the algebra is of type A_1, the result is the classical Clebsch-Gordan Theorem.

THEOREM. *Let π_λ be the irreducible representation with highest weight λ. Then the multiplicity of π_δ in $\pi_\beta \otimes \pi_\gamma$ is*