

the authors allude tantalizingly to the glorious history of what is, after all, the "queen of mathematics." Nevertheless, there is much here that reveals the theory of numbers as a vital and living subject. One way in which this is accomplished is by exploiting the algebraic viewpoint. The beginner is thereby provided with a reservoir of examples to solidify his conception of abstract algebra. There are also topics for the connoisseur as well as the tyro. For example, in Chapter 5 the proof of the theorem of Legendre on the Diophantine equation  $ax^2 + by^2 + cz^2 = 0$  is based upon recent work of Mordell and Skolem. Again, in the last chapter, the proof of Mann's  $\alpha\beta$  Theorem is modeled after the formulation of F. J. Dyson.

It is the reviewer's belief that this elementary text, written specifically for classroom use in American universities, may very well be the best in its class.

A. L. WHITEMAN

*A survey of binary systems.* By R. H. Bruck. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, new series, vol. 20, Berlin-Göttingen-Heidelberg, Springer, 1958. 8+185 pp. DM 36.00.

Except for the vast literature on groups, the subject of binary systems is one that has emerged into the limelight only recently. Suffering in its infancy from being perhaps too much like an appendix to group theory, this book presents irrefutable evidence that this has not been the case for some time. Group theory and also lattices are deliberately and quite appropriately excluded from the survey.

The lion's share of the literature from which this survey is drawn has been discovered during the past two decades, at the same time that related topics from the foundations of geometry, non-associative ring theory and mathematical logic have also received a lot of attention. By now the subject has quite an international following.

This book seems to be the first survey of binary systems. As such it will be invaluable to the expert who desires a quick glance at what has been done as well as an excellent guide to the beginner who seeks an introduction to the subject. We are grateful to the author for having undertaken this study, especially since the quality of the writing is indeed excellent. While virtually every significant result is discussed in the text, some topics are merely mentioned whereas others receive elaborate treatment. In view of the author's many original contributions to the literature it is not surprising that he has chosen to select topics close to home for preferential treatment. Needless to say others might have elected to redistribute the emphasis. For example a proof of Post's result on the unsolvability of the word problem for semi-

groups would have been a welcome addition, especially in view of the more recent progress on the word problem for groups. But this may simply be a matter of taste and also of space and time.

Chapter I, entitled "systems and their generation," deals with binary systems in which the associative law of multiplication is not assumed. The simplest such system is a halfgroupoid, which is merely a set with a binary operation. When the binary operation is defined on the whole set the halfgroupoid becomes a groupoid. Despite the lack of axioms, interesting results about halfgroupoids and groupoids exist, namely those involving the free generation of such systems. Typical are the following: A halfgroupoid freely generates a unique groupoid. Every groupoid is the homomorphic image of some free groupoid. Every sub-halfgroupoid of a free halfgroupoid is free. Every finite or countable halfgroupoid can be imbedded in at least one groupoid with a single generator. These and other results are proved in detail. From here on we are gradually led to quasigroups, loops, nets, projective planes and finally general algebras, the author being careful to preserve the central theme of the chapter. The results on quasigroups are proved, but the remainder of the chapter consists of definitions and brief discussions of the main results.

Chapter II, entitled "the associative law," deals with a variety of systems, including semigroups, Brandt groupoids, polyadic groups and multigroups. A subsection deals with homomorphic imbeddings of groupoids in semigroups. In many instances a discussion of known results coupled with references take the place of proofs or even indications of proofs. However important parts of the structure theory of semigroups as well as Post's coset theorem among other results are presented in detail. The chapter closes appropriately with a schematic diagram showing the interrelation of the systems that have been introduced.

The remaining six chapters, constituting roughly two thirds of the book, are devoted almost entirely to the theory of loops, a subject which owes much to the author's many original contributions. Therein appear the basic concepts and tools as well as the major results of the theory of loops. These derive from the study of isotopy, homomorphisms, loops with operators, direct decompositions, series for loops and entail various notions such as characteristic, normal and Frattini subloops, nilpotency, central nilpotency, refinement of series to mention but a few. Here may be found a lot of material that simply isn't available elsewhere, at least in this latest form of development. The most interesting theorems are perhaps those concerning Moufang loops, to which the last two chapters are devoted. One of the highlights of the book is the proof of the following result.

**THEOREM.** *Every commutative Moufang loop which can be generated by  $n$  elements, where  $n \geq 3$ , is centrally nilpotent of class at most  $n - 1$ .*

This result in fact is so recent that it does not appear elsewhere in the literature.

On the whole the book is highly original. Many of the published results appear in more lucid form. The book is also very readable and appears free of misprints worth mentioning. In other words what it chooses to treat is done in an excellent way. Without a doubt this book is a must for anyone even vaguely interested in the study of loops. To the beginner it offers a survey of the literature on binary systems coupled with an excellent bibliography. It is also a work that is likely to stimulate interest and further research in binary systems.

ERWIN KLEINFELD

*Foundations of geometry, Euclidean and Bolyai-Lobachevskian geometry, projective geometry.* By K. Borsuk and Wanda Szmielew. Revised English translation. Amsterdam, North-Holland Publishing Company, 1960. \$12.00.

Euclid's long lasting popularity is readily explained by the lack of progress on the field covered by the *Elements* during more than twenty centuries although perhaps this sterility was merely a consequence of the authority imposed by the Greek geometrician. The same argument would not suffice to explain the unshaken confidence with which Hilbert's *Grundlagen der Geometrie* is still considered by many people as the definitive revelation of geometric truth. Too much has happened in the last sixty years in the axiomatics of geometry. When I had the opportunity to review the 8th edition of Hilbert's *Grundlagen der Geometrie* in *Nieuw Archief voor Wiskunde*, I tried to answer this question by a detailed analysis of the historical context of that work, and by an appraisal of its positive qualities as well as of its drawbacks. The still overpowering appeal of Hilbert's work is attested anew by the fact that two renowned Polish mathematicians have engaged in the difficult and not too grateful task of elaborating Hilbert's work and adapting it in detail to a more modern concept of mathematics. Though in Euclidean and Lobachevskian geometry Borsuk's and Szmielew's work covers only a small part of Hilbert's booklet, its extent is three or four times that of its predecessor. This proves anew, if ever proof was needed, that the brevity of Hilbert's work was bought by extensive, though mostly minor, omissions. It also proves that Hilbert's lay-out is too complicated and that it can hardly serve as a basis of a simple axiomatic introduction into geom-