A CHARACTERIZATION OF DISCRETE SOLVABLE MATRIX GROUPS

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Introduction. In [1], we introduced a class of solvable groups which we called algebraic strongly torsion free S groups. We will show in this note how these groups can be modified and used to characterize those solvable groups which can be imbedded as discrete subgroups of the group $GL(n, C)$ for some n.

1. **Preliminary discussion and definitions.** Let Γ be a strongly torsion free S group in the sense of H. C. Wang [5]; i.e., Γ satisfies the diagram

$$1 \to D \to \Gamma \to Z^S \to 1$$

where D is a finitely generated torsion free nilpotent group and where Z^S is the additive group of integers taken s times. It is shown in [1] that there exists a unique maximal nilpotent subgroup M of Γ which contains the commutator subgroup $[\Gamma, \Gamma]$. Clearly M is a characteristic subgroup of Γ and torsion free. For any finitely generated torsion free nilpotent group G, we will use $N(G)$ to denote the unique connected simply connected nilpotent Lie group which contains G as a discrete uniform subgroup. We will use $N_C(G)$ to denote the complexification of this Lie group. With this convention made, we may now let $A_1(\Gamma)$ denote the image of Γ in the automorphism group of $N_C(M)$, $A(N_C(M))$, obtained by forming inner automorphisms of Γ. Let $\Gamma^* \subset \Gamma$ be a characteristic subroup of Γ such that Γ/Γ^* is finite, $\Gamma^* \supset M$ and Γ^*/M is torsion free. We may apply the construction of H. C. Wang [5] to the group $S = \Gamma^* N_C(M)$ and obtain $S \subset F \cdot T$, where F is the maximal unipotent subgroup, $F \supset N_C(M)$ as a characteristic subgroup, T is abelian and the dot denotes semi-direct products. We may form $A_1(F) \subset A(N_C(M))$.

DEFINITION. We will say that a strongly torsion free S group is complex algebraic if there exists an abelian analytic group of semisimple elements T^* in $A(N_C(M))$ such that

1. T^* is in the normalizer of $A_1(F)$,
2. $A_1(\Gamma) \subset A_1(F) \cdot T^*$

where the dot denotes the semi-direct product.

REMARK 1. T^* can be considered as an abelian analytic semisimple group of automorphisms of $N_1(F)$, where $N_1(F) \supset F$, $N_1(F)$ is connected simply connected nilpotent Lie group and $N_1(F)/F$ is compact.

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REMARK 2. Γ^* is a discrete subgroup of $N_1(F)\cdot T^*$.

2. Main theorem.

THEOREM. *A necessary and sufficient condition for a solvable group* Γ *to have a faithful discrete matrix representation is that* $\Gamma \supset \Gamma^*$, Γ^* *a complex strongly torsion free S group such that*

1. Γ^* *is normal in* Γ *and* Γ/Γ^* *is finite.*
2. *The group of automorphisms of* Γ^* *induced by inner automorphisms of* Γ *can be extended to* $N_1(F)\cdot T^*$.

PROOF OF SUFFICIENCY. Consider the diagram

$$\begin{array}{ccccccccc} 1 & \to & \Gamma^* & \longrightarrow & \Gamma & \to & \Gamma/\Gamma^* & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & N_1(F)\cdot T^* & \to & S & \to & \Gamma/\Gamma^* & \to & 1 \end{array}$$

Then there exists one and only one group S satisfying this diagram with the action induced from 2 above. By a theorem of Mostow [4], we have $S=EK$, where E is a euclidean space and K is a compact group. Hence the identity component K_0 of K is abelian and $K=K_0\cdot\Gamma/\Gamma^*$ where the dot denotes the semi-direct product. Hence if S_0 is the identity component of S, $S_0=EK_0=N_1(F)\cdot T^*$ and $S=S_0\cdot\Gamma/\Gamma^*$. But $S_0=N_1(F)\cdot T^*$ has a faithful matrix representation by the Birkhoff theorem. Since Γ/Γ^* is finite, this means that S has a faithful matrix representation and hence so does Γ contained in S.

PROOF OF NECESSITY. Let Γ be a discrete solvable subgroup of $GL(n, C)$. Let $H(\ )$ denote the algebraic hull of the group in the bracket and let $H_0(\ )$ denote the identity component of $H(\ )$. Let $\Gamma_1=\Gamma\cap H_0(\Gamma)$. Then Γ_1 is a normal subgroup of Γ and Γ/Γ_1 is finite. Further $H_0(\Gamma)$ is a solvable analytic group. Now let Γ_1^* be a characteristic subgroup of Γ, of finite index all of whose eigen values are 1 or $\cos 2\pi\zeta+i\sin 2\pi\zeta$ where ζ is irrational. Clearly Γ_1^* is normal in Γ, Γ/Γ_1^* is finite and every automorphism of Γ_1^* can be extended to $H(\Gamma_1^*)$. Further Γ^* is clearly a strongly torsion free S group. Let $\Gamma^*=\Gamma_1^*\cap H_0(\Gamma_1^*)$. Then it is trivial to verify that Γ^* satisfies requirements. It is worth noting that one can actually prove that $\Gamma^*=\Gamma_1^*$. Since we do not need this result we will omit it.

REFERENCES

1. L. Auslander, *Discrete uniform subgroups of solvable Lie groups*, Trans. Amer. Math. Soc., to appear.
2. ———, *Discrete solvable matrix groups*, Proc. Amer. Math. Soc. vol. 11 (1960) pp. 687–688.
3. A. Borel, *Groupes linéares algébriques*, Ann. of Math. vol. 64 (1956) pp. 20–82.
4. G. D. Mostow, *Self-adjoint groups*, Ann. of Math. vol. 62 (1955) pp. 44–55.

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