A NEW MAXIMUM PRINCIPLE

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Let $u = u(x)$ be a continuous function satisfying $u \leqq m$ in a region R. A set $K = K_u$ is called "an interior maximum" of u if K is a nonempty component of the set at which $u = m$ and if, furthermore, K is wholly interior to R. Using the notation

$$x = (x_1, x_2, \cdots, x_n), \quad u_i = \partial u/\partial x_i, \quad u_{ij} = \partial^2 u/\partial x_i \partial x_j, \quad u^* = (\sum u_i^2)^{1/2},$$

and similarly for c, we have the following:

Let $a_{ij}(x)$ be a continuous, positive semidefinite matrix function in a bounded region R, and let $g(p)$ be a continuous, positive, increasing function of the real variable $p > 0$. Suppose that $u \in C^{(2)}$ satisfies

$$\sum a_{ij}(x) u_{ij} \geqq -g(u^*), \qquad u \leqq m,$$

and has the interior maximum K_u. For any function $c(x) \in C^{(2)}$ define K_c as the set at which $\sum a_{ij}(x) c_i(x) c_j(x) = 0$. If

$$\int_0 \frac{dp}{g(p)} = \infty,$$

then the sets K_u and K_c must have a point in common.

The condition concerning divergence of the integral is sharp. When K_c is empty the theorem means that K_u cannot exist; in other words, the weak maximum principle holds. The essential meaning in the general case is that, if the maximum principle fails, then the locus $u = m$ cannot be too small.

The proof is obtained by considering a function $w = u + v[c(x)]$ where $v = v_\alpha(s)$ is defined as follows: Let

$$h(p) = Cp + (n^2 U + 1) g(Dp)$$

where C, U and D are constants associated with the bounds of $|a_{ij}|$, $|u_{ij}|$, and c^* in a suitably chosen compact set. We define

$$v = \int_0^s G^{-1}(\alpha - t)\,dt \quad \text{where} \quad G(p) = \int_p^1 \frac{dp}{h(p)}.$$

The same method yields a sharpened form of Hopf's strong maximum principle, viz., if $[a_{ij}(x)]$ is positive definite at each point, and the integral diverges, then the strong maximum principle holds. Extension to discontinuous and unbounded a_{ij} requires but slight additions, the most essential of which is the hypothesis

$$\inf \sum a_{ij}(x) c_{ij}(x) > -\infty.$$

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