ON THE BRAID GROUPS OF E^2 AND S^2

BY EDWARD FADELL[1] AND JAMES VAN BUSKIRK

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1. **Introduction.** The purpose of this paper is to outline a new proof that the geometric braid group on the plane E^2 is isomorphic to the algebraic braid group on E^2 and also to compute the geometric braid group on the sphere S^2. The definition of geometric braid group (see §3) employed is a recent one due to R. H. Fox [3]. It turns out that the geometric braid group on S^2 is just the classical braid group on E^2 with a single additional relation obtained from the simple geometric fact that a simple closed curve on S^2 bounds *two* discs. In [2], it is shown that an associated class of fiber spaces arising naturally from the situation gives information about certain homotopy groups which in turn gives information about the geometric braid groups of certain manifolds. This point of view is pursued in this paper and the central geometric tool here is the fact that the second homotopy group of certain configuration spaces is trivial.

2. **Configuration spaces.** Let M denote a manifold and $x_1, \cdots, x_m$ a fixed set of m mutually distinct points. Set $F_{m,n}(M)$ = the space of n-tuples $(p_1, \cdots, p_n)$ such that $p_i \in M-(x_1 \cup \cdots \cup x_m)$ and $p_i \neq p_j$ if $i \neq j$. Consider $\pi: F_{m,n} \to F_{m,n-r}$, $n > r$, $m \geqq 0$, given by $\pi(p_1, \cdots, p_n) = (p_{r+1}, \cdots, p_n)$.

THEOREM [2]. *$\pi: F_{m,n} \to F_{m,n-r}$ is a locally trivial fiber space with fiber $F_{m+n-r,r}$.*

THEOREM [2]. *If $M = E^2$ (Euclidean 2-space), then $\pi_i(F_{0,n}) = 0$ for $i \geqq 2$, $n \geqq 1$.*

THEOREM. *If $M = S^{k-1}$ ($(k-1)$-sphere), $k \geqq 3$, then the fiber space $\pi: F_{0,3} \to S^{k-1}$ is fiber homotopy equivalent to the bundle $V_{k,2} \to S^{k-1}$, where $V_{k,2}$ is the Stiefel manifold of orthogonal 2-frames in k-space.*

COROLLARY. *If $M = S^2$, $\pi_1(F_{0,3})$ is cyclic of order 2 and $\pi_2(F_{0,n}) = 0$ for $n \geqq 3$.*

3. **The geometric braid groups.** Consider $F_{0,n} = F_{0,n}(M)$, where M is a manifold. Then the full symmetric group Σ^n acts freely on $F_{0,n}$ by permuting coördinates. Let $B_{0,n} = F_{0,n}/\Sigma^n$ and $p: F_{0,n} \to B_{0,n}$ be the associated covering space with fiber Σ^n.

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DEFINITION (R. H. FOX). $\pi_1(B_{0,n})$ is called the *geometric braid group* of M on n strings and is designated by $G_n(M)$.

Let $\gamma: G_n(M)\to\Sigma^n$, with kernel $\pi_1(F_{0,n})$, denote the homomorphism induced by $p: F_{0,n}\to B_{0,n}$. We will also need the following exact sequences. If $M=E^2$ or S^2 the fibering $F_{0,n}\to F_{0,n-1}$ induces exact sequences

$$1 \to \pi_1(F_{n-1,1}) \to \pi_1(F_{0,n}) \to \pi_1(F_{0,n-1}) \to 1$$

where $n\geqq 4$ in case $M=S^2$. This is because $\pi_2(F_{0,n})=0$ (§2).

4. The algebraic braid groups. We recall first the algebraic braid group $B_n(E^2)$ on E^2. $B_n(E^2)$ is the group on $n-1$ generators $\sigma_1, \cdots, \sigma_{n-1}$ subject to the relations

(i) $$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad 1 \leqq i \leqq n-2,$$

(ii) $$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| > 1.$$

Associated with $B_n(E^2)$ is a homomorphism $\alpha: B_n(E^2)\to\Sigma^n$ whose kernel we designate by K_n. Using results of Chow [1], one obtains an exact sequence

$$1 \to A_n \to K_n \to K_{n-1} \to 1$$

where A_n is a certain free subgroup of $B_n(E^2)$.

The algebraic braid group $B_n(S^2)$ on S^2 is obtained by adding a single relation to $B_n(E^2)$. Precisely, $B_n(S^2)$ is the group on $n-1$ generators $\sigma_1, \cdots, \sigma_{n-1}$ subject to the relations

(i) $$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad 1 \leqq i \leqq n-2,$$

(ii) $$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| > 1,$$

(iii) $$\sigma_1\sigma_2 \cdots \sigma_{n-1}\sigma_{n-1} \cdots \sigma_2\sigma_1 = 1.$$

Here too we have an associated homomorphism $\alpha': B_n(S^2)\to\Sigma^n$, whose kernel we designate by K_n', and an exact sequence

$$1 \to A_n' \to K_n' \to K_{n-1}' \to 1$$

where A_n is an appropriate free subgroup of $B_n(S^2)$.

5. The main results. First, there are natural homomorphisms ϕ_n, ϕ_n' giving rise to commutative diagrams

$$\begin{array}{ccc} G_n(E^2) & \xleftarrow{\phi_n} & B_n(E^2) \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\alpha} & \\ & \Sigma^n & \end{array} \qquad\qquad \begin{array}{ccc} G_n(S^2) & \xleftarrow{\phi_n'} & B_n(S^2) \\ & {\scriptstyle\gamma'}\searrow \quad \swarrow{\scriptstyle\alpha'} & \\ & \Sigma^n & \end{array}$$

LEMMA. *ϕ_n is an isomorphism iff $\psi_n = \phi_n | K_n : K_n \to \pi_1(F_{0,n})$ is an isomorphism. The corresponding result is valid in the S^2-case.*

The homomorphisms ψ_n induce a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_n & \longrightarrow & K_n & \longrightarrow & K_{n-1} & \longrightarrow & 1 \\ & & \theta_n \downarrow & & \psi_n \downarrow & & \psi_{n-1} \downarrow & & \\ 1 & \longrightarrow & \pi_1(F_{n-1,1}) & \longrightarrow & \pi_1(F_{0,n}) & \longrightarrow & \pi_1(F_{0,n-1}) & \longrightarrow & 1 \end{array} .$$

Since $\pi_1(F_{n-1,1})$ is also a free group it is quite easy (fortunately the generators correspond precisely under θ_n) to see that θ_n is an isomorphism. It is now clear that induction and the FIVE LEMMA are in order and thus we obtain

THEOREM 1. *$\phi_n : B_n(E^2) \to G_n(E^2)$ is an isomorphism.*

The S^2-case is handled in a similar fashion with a few additional difficulties arising. In this situation one must check the cases $n=2$, 3 separately since $\pi_2(F_{0,n-1})=0$ only if $n \geqq 4$. Thus we obtain

THEOREM 2. *$\phi_n' : B_n(S^2) \to G_n(S^2)$ is an isomorphism.*

REMARK. It may be of interest to remark that although $G_n(E^2)$ has no elements of finite order, $G_n(S^2)$ has elements of finite order for every n. $G_2(S^2)$ and $G_3(S^2)$ are finite groups; the remainder are infinite.

REFERENCES

1. W. Chow, *On the algebraic braid group*, Ann. of Math. vol. 49 (1948) pp. 654–658.
2. E. Fadell and L. Neuwirth, *Configuration spaces*, to appear.
3. R. H. Fox and L. Neuwirth, *The braid groups*, to appear.

THE UNIVERSITY OF WISCONSIN