RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

A PROOF OF THE POWER SERIES EXPANSION WITHOUT CAUCHY'S FORMULA

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In this note a proof will be outlined that a differentiable function of a complex variable has a power series expansion. This work is independent of any integration theory and in particular is independent of Cauchy's formula. These results depend upon one of the basic theorems of topological analysis—namely that if f is differentiable and nonconstant in a region R of the complex plane, then f is an open map, i.e., if O is open in R, $f(O)$ is open in E_2 [5, p. 76].

We adopt the convention that if $f(z)$ is differentiable, the symbol $(f(z)-f(z_0))/(z-z_0)$ represents the function $h(z)=(f(z)-f(z_0))/(z-z_0)$ when $z\neq z_0$ and $h(z)=f'(z_0)$ when $z=z_0$. The function h will be continuous at z_0. A region R is a connected open subset of the complex plane and all functions are from subsets of the complex plane into the complex plane.

LEMMA 1. *Suppose O is a bounded open set and f is continuous on $\bar{O}$ and open on O. Then if W is a complementary domain (component of the complement) of $f(\bar{O}-O)$, $f(O)\cap W\neq\emptyset$ implies $f(O)\supset W$.*

PROOF. Suppose $f(O)\cap W\neq\emptyset$. Then $f(O)\cap W$ is open in W and $f(O)\cap W=f(\bar{O})\cap W$ is closed in W. Since W is connected, $f(O)\supset W$.

LEMMA 2. *Suppose V is open and $p\in V$. If f is continuous on V and open on $V-p$, then f is open on V.*

PROOF. Suppose D is an open set of V containing p. Show that $f(p)$ is in the interior of $f(D)$. Let S be a circle with center p and $S\cup I(S)\subset D$. If $f(p)\in f(S)$ then $f(p)$ will be in the interior of $f(D)$. Assume $f(p)\notin f(S)$ and apply Lemma 1. Let $O=I(S)-p$ and T be the interior of a circle conaining $f(p)$ with $T\cap f(S)=\emptyset$. Note $(T-f(p))\cap f(O)\neq\emptyset$ and that if W is the complementary domain of $f(\bar{O}-O)$ containing $T-f(p)$, $f(O)\supset W\supset T-f(p)$. Therefore $f(\bar{O})\supset T$ and $f(p)$ must be in the interior of $f(\bar{O})$ and thus in the interior of $f(D)$.

The following lemma is the key to the results of this paper. It will be used without reference.

LEMMA 3. *If R is a bounded region, $p \in R$ and h is continuous on $\bar{R}$ and differentiable on $R-p$ (or $R-$ any finite set), then*

$$|h(z)| \leqq \max_{t \in (\bar{R}-R)} |h(t)| \qquad \text{for all } z \in R.$$

PROOF. If h is constant, equality holds. If h is nonconstant, h is open on $R-p$ [5, p. 76] and by Lemma 2 is open on R. The proof is now trivial.

The proof of the following lemma is a simple exercise.

LEMMA 4. *Suppose C is a circle with center z_0 and f is continuous on $C \cup I(C)$. Then if $\epsilon > 0$, $\exists\ \delta > 0$ such that*

$$\left| \frac{f(z_0) - f(z)}{z_0 - z} - \frac{f(y) - f(z)}{y - z} \right| < \epsilon$$

when $z \in C$ and $|y - z_0| < \delta$.

THEOREM 1. *Suppose C is a circle with center z_0. If h is continuous on $C \cup I(C)$ and differentiable on $I(C) - z_0$, then f is differentiable on $I(C)$.*

PROOF. Suppose $\{z_n\}$ is a sequence of points in $I(C)$ which approaches z_0 and $z_i \neq z_0$, $i = 1, 2, 3, \cdots$. It will be shown that the sequence $(f(z_0) - f(z_n))/(z_0 - z_n)$ is Cauchy. Let $\epsilon > 0$. By a double application of Lemma 4 and the triangle inequality, there exists an N such that

$$\left| \frac{f(x) - f(z_n)}{x - z_n} - \frac{f(x) - f(z_m)}{x - z_m} \right| < \epsilon$$

for all $x \in C$ whenever n and $m \geqq N$. Let n and m be integers each $\geqq N$. Then

$$h(z) = \frac{f(z) - f(z_n)}{z - z_n} - \frac{f(z) - f(z_m)}{z - z_m}$$

is continuous on $C \cup I(C)$ and differentiable on $I(C) - (z_n + z_m + z_0)$. Thus by Lemma 3, the maximum modulus theorem holds and

$$|h(z)| < \epsilon \qquad \text{for all } z \text{ in } I(C).$$

In particular,

$$|h(z_0)| = \left| \frac{f(z_0) - f(z_n)}{z_0 - z_n} - \frac{f(z_0) - f(z_m)}{z_0 - z_m} \right| < \epsilon.$$

Thus $(f(z_0)-f(z_n))/(z_0-z_n)$ is a Cauchy sequence and therefore converges. Let $\{z_n'\}$ be another sequence converging to z_0. The corresponding difference quotients must again converge. To see that the limit is the same, consider the sequence $z_1, z_1', z_2, z_2', \cdots$. Thus f is differentiable at z_0.

COROLLARY (not necessary for the power series development). *Suppose C is a circle with center z_0 and f is continuous and bounded on $C\cup I(C)-z_0$ and differentiable on $I(C)-z_0$. Then f can be defined at z_0 such that it will be continuous and differentiable there.*

PROOF. Let $h(z)=(z-z_0)f(z)$ when $z\neq z_0$ and $h(z)=0$ when $z=z_0$. Then $h(z)$ will be continuous on $C\cup I(C)$ and differentiable on $I(C)-z_0$. By Theorem 1, h is differentiable at z_0, i.e.,

$$\lim_{z\to z_0}\frac{h(z)-h(z_0)}{z-z_0}$$

exists. Since $(h(z)-h(z_0))/(z-z_0)=f(z)$, $\lim_{z\to z_0}f(z)$ exists. Thus f can be defined at z_0 so that it will be continuous there. Now by Theorem 1, $f(z)$ will also be differentiable at z_0.

The following theorem can be proved by the same simple procedures demonstrated here (see forthcoming paper in Duke Math. J.). However, its proof will be omitted to allow space for the power series development.

THEOREM 2. *If f is differentiable in a region R, then f' is differentiable in R.*

LEMMA 5. *Suppose R is a region containing the origin and n is a positive integer. If f is differentiable in R and*

$$f(0)=f^1(0)=\cdots=f^{n-1}(0)=0,\quad\text{then}\quad \lim_{z\to 0}\frac{f(z)}{z^n}=\frac{f^n(0)}{n!}\cdot$$

PROOF. The theorem is true for $n=1$. Suppose the theorem holds for $n=N$ and f satisfies the hypothesis for $n=N+1$. Then $\lim_{z\to 0}f(z)/z^N=f(0)/N!=0$. By Theorem 1, $h(z)=f(z)/z^N$, $h(0)=0$ is differentiable in R. Thus

$$\lim_{z\to 0}\frac{h(z)-h(0)}{z-0}=\lim\frac{f(z)}{z^{N+1}}$$

exists and is $h^1(0)$. But

$$h^1(0)=\lim_{z\to 0}h^1(z)=\lim_{z\to 0}\frac{f^1(z)}{z^N}-\frac{Nf(z)}{z^{N+1}}\cdot$$

Thus

$$\lim_{z\to 0}\frac{f(z)}{z^{N+1}}=\frac{1}{N+1}\lim_{z\to 0}\frac{f^1(z)}{z^N},$$

which is $f^{N+1}(0)/(N+1)!$ by the inductive hypothesis.

THEOREM 3. *Suppose R is the interior of the unit circle and $f\colon \overline{R}\to E_2$ is continuous on $\overline{R}$ and differentiable on R, and $|f(z)|\leqq 1$ for z in R. Then $|f^n(0)/n!|\leqq 2^n$ for all n and the Taylor series for f converges to f for $|z|<1/2$.*

PROOF. Let $h_1(z)=f(z)-f(0)$. Then $|f^1(0)|\leqq\max_{t\in(\overline{R}-R)}|h_1(t)/t| =\max_{t\in(\overline{R}-R)}|h_1(t)|\leqq 2$. Suppose $|f^n(0)/n!|\leqq 2^n$ for $n=1, 2, 3, \cdots (N-1)$. Let

$$h_N(z)=f(z)-f(0)-\left(\frac{f^1(0)}{1!}\right)z-\left(\frac{f^2(0)}{2!}\right)z^2-\cdots-\left(\frac{f^{N-1}(0)}{(N-1)!}\right)z^{N-1}.$$

Then $h_N^i(0)=0$, $i=1, 2, \cdots, N-1$. By Lemma 5, $\lim_{z\to 0} h_N(z)/z^N =h_N^N(0)/N!$ which is $f^N(0)/N!$. Thus

$$\left|\frac{f^N(0)}{N!}\right|\leqq\max_{t\in(\overline{R}-R)}\left|\frac{h_N(t)}{t^N}\right|,$$

which is $\leqq 1+1+2+2^2+\cdots+2^{N-1}=2^N$ by the inductive hypothesis. For the convergence of the power series, note that

$$\left|\frac{h_N(z)}{z^N}\right|\leqq\max_{t\in(\overline{R}-R)}\left|\frac{h_N(t)}{t^N}\right|\leqq 2^N.$$

Thus $|h_N(z)|\leqq|2z|^N$ and $h_N(z)$ approaches zero for $|z|<1/2$.

A short continuation of this approach also yields the Laurent expansion. Other classical theorems such as "entire bounded functions are constant" and "the uniform limit of differentiable functions is differentiable" follow directly from Lemma 3.

There is an alternate proof of the power series expansion which does not require the intermediate step of proving all the derivatives exist (Theorem 2). This approach is a slight variation of an algorithm of J. Schur [3] and uses products of linear fractional transformations. It is based upon the maximum modulus theorem, the removable singularity theorem, and an algebraic technique for estimating bounds on the coefficients of a power series [4]. This constructive procedure yields the full radius of convergence and will be published at a later date.

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