RESEARCH PROBLEMS

1. Richard Bellman: *Generalized exponentials and Baker-Hausdorff-Campbell series.*

Consider the equation

$$u = 1 + T(xu), \tag{1}$$

where T is a linear transformation and x and u are elements of a suitable space. As pointed out by Baxter, if T enjoys the following generalized "integration by parts" formula

$$T(xT(y)) + T(yT(x)) = T(x)T(y) \tag{2}$$

then $u = e^{T(x)}$ is a solution of (1), and the solution under appropriate assumptions.

Generally, let us write $u = E(x)$ to denote the solution of (1), a *generalized exponential,* and introduce a *generalized bracket symbol*

$$[x, y; T] = T(xT(y)) + T(yT(x)) - T(x)T(y). \tag{3}$$

It is easy to verify that this enjoys the Friedrichs-Magnus property

$$[x + x', y + y'; T] = [x, y; T] + [x', y'; T]. \tag{4}$$

One would suspect in view of the foregoing remarks that this new bracket symbol plays the same role in the study of the function $E(x)$ that the classical commutator, $[A, B] = AB - BA$, plays in the study of the matrix exponential.

We would thus expect formulas of the type

$$E(x + y) = E(x)E(y)E([x, y; T]) \cdots, \tag{5}$$

where the further terms contain iterations of the bracket operation, and equivalently, a generalized Baker-Hausdorff-Campbell formula

$$E(x)E(y) = E(x + v + [x, y; T]/2 + \cdots). \tag{6}$$

Finally, there should be an associated generalized Lie algebra.

Do formulas of the above type exist, and how does one obtain them?

2. Robert J. Aumann: *Extending an order.*

Let Z^n denote the set of lattice points in euclidean n-space, I^n the set of points in Z^n all of whose coordinates are 0 or 1 (i.e., the vertices of the unit cube). Let $>$ be a total order on I^n which is "consistent" in the sense that for $x, y, z, x+z, y+z \in I^n$, $x > y$ implies $x+z > y+z$ (ordinary vector addition is meant). Is it always possible to extend $>$ to a "consistent" order on Z^n?