ON THE DETERMINANTS OF CERTAIN TOEPLITZ MATRICES

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With a function of the form

$$\phi(\theta) + g(x) = \sum_{\nu=-\infty}^{\infty} a_\nu e^{i\nu\theta} + g(x), \qquad 0 \leqq x \leqq 1,$$

we associate for each $n=0, 1, 2, \cdots$, a Toeplitz matrix

$$T_n(\phi(\theta) + g(x)) = \left\{ a_{i-j} + \delta_{ij} g\left(\frac{i}{n+1}\right)\right\}, \qquad i, j = 0, 1, \cdots, n,$$

where $\delta_{ij}=1$ if $i=j$, $\delta_{ij}=0$ if $i\neq j$.

Furthermore we define

$$D_n(\phi(\theta) + g(x)) = \det T_n(\phi(\theta) + g(x)),$$

$$G(\phi(\theta) + g(x)) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 \log(\phi(\theta) + g(x))dxd\theta\right\},$$

$$L(\phi(\theta) + g(x)) = \lim_{n\to\infty} \frac{D_n(\phi(\theta) + g(x))}{[G(\phi(\theta) + g(x))]^{n+1}},$$

whenever these definitions make sense.

We shall prove that under the conditions

(i) $g(x)$ is real and differentiable for $0\leqq x\leqq 1$ with $g'(x)$ satisfying the Lipschitz condition

$$|g'(x_1) - g'(x_2)| < K|x_1 - x_2|^\alpha, \qquad K > 0, \alpha > 0,$$

(ii) $\phi(\theta)$ is a trigonometric polynomial of the type

$$\phi(\theta) = \sum_{\nu=-k}^{k} a_\nu e^{i\nu\theta},$$

$$a_0 = 0,\ a_\nu = a_{-\nu},\ a_\nu \text{ real}, \qquad \nu = 1, 2, \cdots, k,$$

(iii) $$\sum_{\nu=-k}^{k} |a_\nu| < g(x), \qquad \text{for } 0 \leqq x \leqq 1,$$

the limit $L(\phi(\theta)+g(x))$ exists and has the value

$$L(\phi(\theta) + g(x)) \tag{1}$$
$$= \left(\frac{G(\phi(\theta) + g(0))}{G(\phi(\theta) + g(1))} L(\phi(\theta) + g(0))L(\phi(\theta) + g(1))\right)^{1/2}.$$

(For the existence and the value of $L(\phi(\theta)+g(0))$ and $L(\phi(\theta)+g(1))$, see [1, Theorem 5.5, p. 76] or [2].)

Without loss of generality we may assume that

$$\sum_{\nu=-k}^{k} |a_\nu| + g(x) < 1, \qquad 0 \leqq x \leqq 1.$$

Let

$$|\phi|(\theta) = \sum_{\nu=-k}^{k} |a_\nu| e^{i\nu\theta}.$$

We then have

$$0 < -\phi(\theta) + 1 - g(x) < \beta < 1,$$
$$0 < |\phi|(\theta) + 1 - g(x) < \beta < 1,$$

for $-\pi \leqq \theta \leqq \pi$, $0 \leqq x \leqq 1$.

Let us introduce

$$H_n(\phi(\theta) + g(x)) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{\mu=0}^{n} \log\left(\phi(\theta) + g\left(\frac{\mu}{n+1}\right)\right) d\theta\right\}.$$

By use of (i) it is easy to prove that

$$\text{(2)} \qquad \lim_{n\to\infty} \frac{H_n(\phi(\theta) + g(x))}{[G(\phi(\theta) + g(x))]^{n+1}} = \left(\frac{G(\phi(\theta) + g(0))}{G(\phi(\theta) + g(1))}\right)^{1/2}.$$

Now we have

$$\log D_n(\phi(\theta) + g(x)) = -\sum_{p=1}^{\infty} \frac{1}{p} \operatorname{Tr}\{T_n^p(-\phi(\theta) + 1 - g(x))\},$$

$$\log H_n(\phi(\theta) + g(x)) = -\sum_{p=1}^{\infty} \frac{1}{p} \sum_{\mu=0}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(-\phi(\theta) + 1 - g\left(\frac{\mu}{n+1}\right)\right)^p d\theta$$

and hence

$$\text{(3)} \qquad \log \frac{D_n(\phi(\theta) + g(x))}{H_n(\phi(\theta) + g(x))} = \sum_{p=1}^{\infty} \frac{1}{p} \sum_{\mu=0}^{n} E_\mu(n, p),$$

where

$$E_\mu(n, p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(-\phi(\theta) + 1 - g\left(\frac{\mu}{n+1}\right)\right)^p d\theta - \{T_n^p(-\phi(\theta) + 1 - g(x))\}_{\mu\mu}.$$

Put $\gamma = \max_{0 \leqq x \leqq 1} (1 - g(x))$. Then

$$(4) \qquad | E_\mu(n, p) | \leqq \beta^p + \frac{1}{2\pi}\int_{-\pi}^{\pi} (| \phi | (\theta) + \gamma)^p d\theta \leqq 2\beta^p,$$

for $n=0, 1, 2, \cdots; p=1, 2, \cdots; \mu=0, 1, 2, \cdots, n$. For $n \geqq 2pk$ and $pk \leqq \mu \leqq n-pk$ we have

$$E_\mu(n, p) = \sum_{\alpha_1+\cdots+\alpha_p=0} \prod_{\alpha_j\neq 0} (-a_{\alpha_j})$$

$$\cdot\left[\prod_{\alpha_j=0}\left(1-g\left(\frac{\mu}{n+1}\right)\right) - \prod_{\alpha_j=0}\left(1-g\left(\frac{\mu+\alpha_1+\cdots+\alpha_j}{n+1}\right)\right)\right]$$

from which we get

$$| E_\mu(n, p) | \leqq \text{constant}\,\frac{p^2k}{n+1}\sum_{\alpha_1+\cdots+\alpha_p=0}\prod_{\alpha_j\neq 0} | a_{\alpha_j} | \prod_{\alpha_j=0}\gamma,$$

where the constant only depends on $g(x)$ and $g'(x)$. Hence

$$(5) \qquad \begin{aligned} | E_\mu(n, p) | &\leqq \text{constant}\,\frac{p^2k}{n+1}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}(| \phi | (\theta) + \gamma)^p d\theta \\ &\leqq \text{constant}\,\frac{p^2k}{n+1}\,\beta^p, \end{aligned}$$

for $n \geqq 2pk$ and $pk \leqq \mu \leqq n-pk$. By combining (4) and (5) we see that (3) is dominated by a convergent series of the form

$$\text{constant} \sum_{p=1}^{\infty} p\beta^p.$$

To get a better estimate of $E_\mu(n, p)$ for $n \geqq 2pk$ and $pk \leqq \mu \leqq n-pk$ we let $\sum^*$ denote summation over all $(\alpha_1, \alpha_2, \cdots, \alpha_p) \neq (0, 0, \cdots, 0)$ satisfying

$$\alpha_j = 0, \pm 1, \pm 2, \cdots, \pm k, \qquad j = 1, 2, \cdots, p,$$
$$\alpha_1 + \alpha_2 + \cdots + \alpha_p = 0,$$
$$\text{first } \alpha_j \neq 0 \text{ is positive.}$$

Then

$$\begin{aligned} E_\mu(n, p) = \sum{}^* \prod_{\alpha_j\neq 0}(-a_{\alpha_j})\Bigg[2\prod_{\alpha_j=0}\left(1-g\left(\frac{\mu}{n+1}\right)\right) &- \prod_{\alpha_j=0}\left(1-g\left(\frac{\mu+\alpha_1+\cdots+\alpha_j}{n+1}\right)\right) \\ &- \prod_{\alpha_j=0}\left(1-g\left(\frac{\mu-\alpha_1-\cdots-\alpha_j}{n+1}\right)\right)\Bigg]. \end{aligned}$$

From (i) it follows that

$$g(x+\Delta x) = g(x) + g'(x)\Delta x + R(x, \Delta x)\Delta x^{1+\alpha},$$

where $R(x, \Delta x)$ is bounded. Using this we get

$$|E_\mu(n, p)| \leqq \text{constant}\left(\frac{1}{n+1}\right)^{1+\alpha},$$

where the constant does not depend on n. Hence

$$\lim_{n\to\infty} \sum_{\mu=pk}^{n-pk} E_\mu(n, p) = 0,$$

and therefore

$$\lim_{n\to\infty} \sum_{\mu=0}^{n} E_\mu(n, p) = \sum_{\mu=0}^{pk-1} u(\mu, p) + \sum_{\mu=0}^{pk-1} v(\mu, p), \tag{6}$$

where

$$u(\mu, p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (-\phi(\theta) + 1 - g(0))^p \, d\theta - \{T_{2pk}^p(-\phi(\theta) + 1 - g(0))\}_{\mu\mu},$$

$$v(\mu, p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (-\phi(\theta) + 1 - g(1))^p d\theta - \{T_{2pk}^p(-\phi(\theta) + 1 - g(1))\}_{\mu\mu}.$$

From (6) we conclude that

$$\begin{aligned} \lim_{n\to\infty} \sum_{p=1}^{\infty} \frac{1}{p} \sum_{\mu=0}^{n} E_\mu(n, p) &= \sum_{p=1}^{\infty} \frac{1}{p} \sum_{\mu=0}^{pk-1} u(\mu, p) + \sum_{p=1}^{\infty} \frac{1}{p} \sum_{\mu=0}^{pk-1} v(\mu, p) \\ &= \frac{1}{2} \log L(\phi(\theta) + g(0)) + \frac{1}{2} \log L(\phi(\theta) + g(1)) \end{aligned} \tag{7}$$

(the last equality being the result of a straightforward computation).

The formula (1) now follows by combining (2), (3), and (7).

References

1. U. Grenander and G. Szegö, *Toeplitz forms and their applications*, University of California Press, 1958.

2. G. Szegö, *On certain hermitian forms associated with the Fourier series of a positive function*, Festskrift Marcel Riesz, Lund, 1952.

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