A REMARK ON PICARD'S THEOREM

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Communicated by P. R. Garabedian, October 3, 1960

1. In a recent manuscript K. Matsumoto has proved that there exists no general Picard theorem for functions meromorphic outside a set of logarithmic capacity zero. More precisely: given a closed set E of logarithmic capacity zero, then there exists another closed set F of capacity zero and a function $f(z)$, meromorphic in the complement of F and with essential singularities at all points of F, such that $f(z)$ omits the set E. This result indicates that there is no Picard theorem at all for any perfect set. We shall here show that there still is an essential problem left, by giving an example of a set F for which a strong Picard's theorem holds.[1]

THEOREM. *There exists a linear closed set of positive capacity such that every function $f(z)$, meromorphic outside F and omitting 4 values is rational.*

2. We construct F as a Cantor set on $(0, 1)$ where the successive ratios ξ_ν decrease and satisfy the condition

$$\lim_{\nu=\infty} \frac{\log \xi_\nu}{\log \nu} = -\infty. \tag{2.1}$$

Since F is of positive capacity if and only if

$$\sum_1^\infty \frac{\log \xi_\nu^{-1}}{2^\nu} < \infty,$$

ξ_ν can be chosen so that cap $(F)>0$. Let $f(z)$ be meromorphic outside F and assume that $f(z) \neq a_1, a_2, a_3, a_4$. Define $f_\nu = (1-fa_\nu)(f-a_\nu)^{-1}$.

For the functions f_ν, the following lemmas hold. M denotes constants only depending on a_ν.

LEMMA 1. *If f_ν is holomorphic in $\rho \leqq |z-a| \leqq 2\rho$ and $|f_\nu(z_0)| \leqq M$, $|z_0-a| = 3\rho/2$, then $|f_\nu(z)| \leqq M$ for all z, $9\rho/8 \leqq |z-a| \leqq 15\rho/8$.*

PROOF. This is Schottky's theorem.

LEMMA 2. *If $f_\nu(z)$ is holomorphic in $\rho \leqq |z-a| \leqq K\rho$, $K \geqq 2$, and $|f_\nu(z)|$ is $<M$, then the circle $|z-a| = K^{1/2}\rho$ is mapped on a set of diameter $<MK^{-1/2}$.*

[1] This type of strong Picard theorem was used by O. Lehto, *A generalization of Picard's theorem*, Ark. Mat. vol. 3 (1958) p. 495.

PROOF. We consider $g(z)=f(\rho(z-a))$ in $1\leqq|z|\leqq K$ and estimate $g'(z)$ on $|z|=K^{1/2}$ by means of the Cauchy integral.

3. Let I_{n-1} be an interval in the $(n-1)$st subdivision of (0, 1). In the next construction, I_{n-1} is divided into I_n and I_n' of lengths l_n and a complementary interval ω_n. With the midpoint of I_n as center, we construct the following circles:

$$C_n \text{ with radius } l_n,$$

$$\Gamma_n \text{ with radius } \frac{1}{3}\, l_{n-1},$$

$$\gamma_n \text{ with radius } \left(\frac{1}{3}\, l_n l_{n-1}\right)^{1/2},$$

C_n', Γ_n', γ_n' are the corresponding circles for I_n'.

By Lemma 1 we have for, e.g., $\nu=1, 2, 3$,

$$|f_\nu(z)| < M, \qquad z\in\Gamma_n,\Gamma_n' \qquad (\nu = 1, 2, 3) \tag{3.1}$$

and for at least two indices

$$|f_\nu(z)| < M, \qquad z\in C_n \qquad (\text{e.g., } \nu = 1, 2)$$

and

$$|f_\nu(z)| < M, \qquad z\in C_n' \qquad (\text{e.g., } \nu = 1, 3).$$

According to Lemma 2 there exist complex numbers b and b', $|b| < M$, $|b'| < M$, such that

$$|f_1 - b| < M/(K_n)^{1/2} \text{ on } \gamma_n \tag{3.2}$$

and

$$|f_1 - b'| < M/(K_n)^{1/2} \text{ on } \gamma_n', \tag{3.3}$$

where

$$K_n = \left(\frac{l_{n-1}}{l_n}\right)^{1/2} = \xi_n^{-1/2}.$$

We now assume that

$$|f_1 - a| < \epsilon_{n-1} \text{ on } C_{n-1}$$

and consider the function

$$g(z) = (f_1 - a)(f_1 - b)(f_1 - b'),$$

in the region R bounded by C_{n-1}, γ_n and γ_n'. On the boundary of R,

$$|g(z)| < M\epsilon_{n-1} + M/(K_n)^{1/2}$$

which, by the maximum principle implies

$$|f_1 - a| < M(\epsilon_{n-1})^{1/3} + M/(K_n)^{1/6} \tag{3.4}$$

in the region R. The same inequality holds for b and b'.

One function, e.g., $f_2(z)$, is bounded on γ_n and the C-curves C_{n+1} and C_{n+1}' inside γ_n. It follows from (3.2), (3.3) and (3.4) that there exists c so that

$$|f_2 - c| < M/(K_n)^{1/2} \text{ on } \gamma_n. \tag{3.5}$$

If $\omega_n(z)$ is harmonic between $\gamma_n C_{n+1}'$ and C_{n+1}, $\omega_n = 1$ on γ_n and $\omega_n = 0$ on C_{n+1}' and C_{n+1}, an explicit calculation shows that $\omega_n > 1/M$ on C_n since ξ_ν decreases. It follows from (3.5) that

$$\log |f_2 - c| < -\frac{\log K_n}{M} \text{ on } C_n.$$

Hence it follows that for certain α_ν and a fixed $\eta > 0$

$$[f_\nu, \alpha_\nu] \leqq K_n^{-\eta} \text{ on } C_n,$$

where [,] denotes cordal distance. Further $\epsilon_n < MK_n^{-\eta}$. Since $\sum K_n^{-\eta} < \infty$ for all $\eta > 0$, it follows that $f(z)$ is continuous as a function to the Riemann sphere. Hence $f(z)$ is rational, since F is removable for bounded holomorphic functions.

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