DIMENSION OF THE SQUARE OF A SPACE

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In [1] a two-dimensional, compact, metric space B is constructed, whose square $B\times B$ is three-dimensional, instead of being four-dimensional. Hence, for every $n\geqq 2$, there are spaces X of dimension n, such that $\dim(X\times X)\leqq 2n-1$ (take, for example, $X=B\times S^{n-2}$). However, Boltyanskiĭ's example is the best possible, as we will prove that the last inequality cannot be improved, at least in the class of spaces for which dimension theory is the most meaningful [3, p. 153].

THEOREM 1. *Let X be a locally compact, separable, metric space of dimension $\geqq n$. Then* $\dim(X\times X)\geqq 2n-1$; *equality may hold true.*

We conjecture that Theorem 1 remains true for separable, metric spaces, but we do not know whether the proof below applies or not.

Besides the usual cohomology technique, i.e., Künneth's exact sequence, the proof of Theorem 1 is based on a purely algebraic remark (see Theorem 2 below). It is well known that for *finite* abelian groups $A\otimes A\cong \operatorname{Tor}(A, A)$, but the isomorphism is not natural ($\otimes$ and Tor are taken over the ring of integers Z). Curiously enough, the proof of our topological theorem is based on the fact that this remark is far from being true for infinite torsion groups.

THEOREM 2. *If A is a nonzero abelian group such that $A\otimes A=0$, then* $\operatorname{Tor}(A, A)\neq 0$.

PROOF OF THEOREM 2. Let A be an abelian group, such that $A\otimes A=0$. It is not difficult to prove then,[1] that A is a completely divisible torsion group. By a well known theorem of Prüfer [4, p. 165], a completely divisible torsion group is the direct sum of so called Prüfer groups P. (In [4, p. 163], completely divisible groups are called complete; p-primary Prüfer groups are termed p^{∞}-groups.) A p-primary Prüfer group P is $\cong P'/Z$, where P' is the additive group of those rationals whose denominator is a power of p. Hence, given A, as in Theorem 2, there is a set I, and for every $i\in I$ a prime p_i, and a p_i-primary Prüfer group P_i, such that

[1] This statement, and similar ones below, can easily be deduced from results of [2]. A more detailed version of the proofs of this paper, as well as some discussion of the machinery involved, appears as Technical Report, under the title *Dimension of the square of a space.* II, and is available from the Department of Mathematics, University of California, Berkeley 4, California.

$$A = \sum_{i \in I} P_i. \tag{1}$$

Now Tor(B, C) is an additive functor in each variable. If B is p-primary, and C is q-primary, $p \neq q$, then Tor$(B, C) = 0$. It is also easy to prove that Tor$(P, P) \cong P$ for a p-primary Prüfer group P, this isomorphism being non-natural. Thus (1) implies

$$\mathrm{Tor}(A, A) = \sum_{(i,j)} P_i,$$

where the direct sum is to be taken over all $(i, j) \in I \times I$ for which $p_i = p_j$. In particular, this direct sum contains a partial sum identical to the right hand side of (1). This shows that we have a non-natural inclusion

$$\mathrm{Tor}(A, A) \supset A,$$

from which the theorem follows immediately.

Proof of Theorem 1. If X is a compact space, we denote by $HX = H(X) = \sum_p H^p(X)$ the Čech cohomology ring of X with *integer coefficients*. If X is locally compact, for example, if it is an open sub-space of a compact space, then HX stands for the *compact cohomology*. Thus $HU \to HX \to HC \to HU$ is an exact sequence, if $U = X - C$, and C is closed in X.

Given the locally compact spaces X, Y, the Künneth exact sequence is then

$$0 \to HX \otimes HY \xrightarrow{i} H(X \times Y) \xrightarrow{j} \mathrm{Tor}(HX, HY) \to 0, \tag{2}$$

where all groups are bi-graded, i conserves both grades and j increases the total grade by 1 (see, for example, [**5**, p. 255]).

From dimension theory, we use the well known fact [**3**, p. 152] that $\dim X = n$ if and only if, $\dim X < \infty$, the cohomology of every sub-space of X is trivial in dimensions $\geqq n+1$, and there exists an *open* sub-space U in X such that $H^n(U) \neq 0$. Here X is supposed to be locally compact, separable, metric.

In order to prove Theorem 1, let us consider a locally compact space X, such that $\dim X = n$, $\dim(X \times X) \leqq 2n - 1$. Let us choose an open sub-space U of X, for which $H^n(U) \neq 0$. As $U \times U$ is an open sub-space of $X \times X$, $H^{2n}(U \times U) = 0$, by our hypothesis. Applying (2) in dimension $2n$, we get $H^n(U) \otimes H^n(U) = 0$. From Theorem 2 follows then, that Tor$(H^n(U), H^n(U)) \neq 0$. We use now (2) in one dimension less, that is

$$H^{2n-1}(U \times U) \to \operatorname{Tor}(H^n(U), H^n(U)) \to 0.$$

Exactness of this sequence implies that the first group is nonzero, which shows that $\dim(X \times X) \geqq 2n-1$. This completes the proof of the theorem.

Bibliography

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