THE CHARACTERIZATION OF FUNCTIONS ARISING AS POTENTIALS

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1. Introduction. The purpose of this note is to announce results giving the characterization of classes of functions arising by fractional integration. We consider two fractional integral operators I_α and J_α, defined for a suitable class of functions on E_n as follows:

$$I_\alpha(f)^\frown(x) = |x|^{-\alpha} f^\frown(x), \qquad 0 < \alpha < n,$$
$$J_\alpha(f)^\frown(x) = (1 + |x|^2)^{-\alpha/2} f^\frown(x), \qquad 0 < \alpha.$$

The symbol $^\frown$ denotes the Fourier transform.

The integral I_α is a well-known Riesz potential, while the integral J_α is a modification of it, the so-called "Bessel potential." The local behavior of I_α and J_α are equivalent, but the global behavior of J_α is more tractable since $J_\alpha f = K_\alpha^* f$, where $K_\alpha \geqq 0$, and $K_\alpha \in L^1(E_n)$.

We denote by L_α^p the class of functions f of the form $f = J_\alpha(\phi) = K_\alpha^* \phi$, where $\phi \in L^p(E_n)$. We shall always make the restriction $1 < p < \infty$. The classes L_α^p and the operators J_α have been studied by several authors; see [1] for the L_2 theory, and [2] for the L^p theory. We seek to characterize the functions $f \in L_\alpha^p$ in terms of their "smoothness," i.e., in terms of the smallness of $f(x+y) - f(x)$.

We recall first a useful fact. If $f \in L_\alpha^p$, $\alpha \geqq 1$, then $f \in L_{\alpha-1}^p$ and $\partial f/\partial x_k \in L_{\alpha-1}^p$, $k = 1, \cdots, n$, and conversely. Thus in many cases it is sufficient to restrict our attention to $0 < \alpha < 1$.

Our characterization will be in terms of the "functional" $\mathfrak{D}_\alpha$

$$(1) \qquad \mathfrak{D}_\alpha(f)(x) = \left(\int_{E_n} \frac{|f(x-y) - f(x)|^2}{|y|^{n+2\alpha}} \, dy \right)^{1/2}, \qquad 0 < \alpha < 1,$$

and its variants.

2. Main results.

THEOREM 1. *Let* $0 < \alpha < 1$, $2n/(n+2\alpha) < p < \infty$. *Then* $f \in L_\alpha^p$ *(i.e.,* $f = J_\alpha(\phi)$, $\phi \in L^p$*) if and only if* (a) $f \in L^p$, *and* (b) $\mathfrak{D}_\alpha(f) \in L^p$. *Also*

$$B_{\alpha,p} \|\phi\|_p \leqq \|\mathfrak{D}_\alpha(f)\|_p + \|f\|_p \leqq A_{\alpha,p} \|\phi\|_p.$$

REMARKS. (i) The restriction $2n/(n+2\alpha) < p$ is essentially necessary. If $p < 2n/(n+2\alpha)$, there exists $\phi \in L^p$, so that $f = J_\alpha(\phi)$ is not locally in L^2 and so that $\mathfrak{D}_\alpha(f)(x) = \infty$, all x.

(ii) We can obtain results analogous to Theorem 1 by replacing the functional $\mathfrak{D}_\alpha$ by either $\mathfrak{D}_\alpha^{(1)}$ or by $\mathfrak{D}_\alpha^{(2)}$ defined by

$$\mathfrak{D}_\alpha^{(1)}(f)(x) = \left(\int_{E_n} \frac{|f(x-y) - f(x+y)|^2}{|y|^{n+2\alpha}} dy\right)^{1/2},$$

$$\mathfrak{D}_\alpha^{(2)}(f)(x) = \left(\int_{E_n} \frac{|f(x+y) + f(x-y) - 2f(x)|^2}{|y|^{n+2\alpha}} dy\right)^{1/2}.$$

In the case $\mathfrak{D}_\alpha^{(1)}$, and when $n=1$, we recover essentially some of the results of Hirschman [**4**] (see also Flett [**3**]), which results were the starting point of this investigation. The results for $\mathfrak{D}_\alpha^{(2)}$ hold, in fact, for the wider range $0<\alpha<2$. In this instance the special case $\alpha=1$ may be viewed as another generalization of the integral of Marcinkiewicz to several variables.

(iii) The variants of Theorem 1 which hold for all p, $1<p<\infty$, are closely related to the generalization of the integral of Marcinkiewicz discussed in §8 of [**6**]. These will be treated elsewhere.

We outline the proof of Theorem 1. The idea is to reduce the problem to one dealing with the functions g^* and g, considered previously (see [**5**; **6**]). That such a reduction might be possible is indicated by the one-dimensional case treated by Hirschman.

LEMMA 1. *Let ϕ be bounded and vanish outside a bounded set. Let $\phi_\alpha = I_\alpha(\phi)$. Then if $0<\alpha<1$, $2n/(n+2\alpha)<p<\infty$,*

$$B_{p,\alpha}\|\phi\|_p \leqq \|\mathfrak{D}_\alpha(\phi_\alpha)\|_p \leqq A_{p,\alpha}\|\phi\|_p.$$

This lemma is a consequence of another two:

LEMMA 2. $\mathfrak{D}_\alpha(\phi_\alpha)(x) \leqq A_{\lambda,\alpha} g_\lambda^*(x;\phi)$, $0<\lambda<2\alpha$, $0<\alpha<1$.

LEMMA 3. $B_\alpha g(x;\phi) \leqq \mathfrak{D}_\alpha(\phi_\alpha)(x)$, $0<\alpha<1$.

A combination of Lemmas 2 and 3, together with known facts about g_λ^* and g (see [**5**; **6**]) proves Lemma 1.

The theorem then follows from Lemma 1 and the following lemma which relates the operators I_α and J_α.

LEMMA 4. *Let $0<\alpha$. There exists finite measures on E_n, $d\mu_\alpha^{(1)}$, $d\mu_\alpha^{(2)}$, and $d\mu_\alpha^{(3)}$, so that if $F_\alpha^{(i)}(x)$ are the Fourier transforms of $d\mu_\alpha^{(i)}$, then*

$$(1+|x|^2)^{\alpha/2} = F_\alpha^{(1)}(x) + F_\alpha^{(2)}(x)\cdot|x|^\alpha,$$
$$|x|^\alpha = F_\alpha^{(3)}(x)\cdot(1+|x|^2)^{\alpha/2}.$$

3. **Another characterization.** We shall now consider another characterization of the functions of class L_α^p, which is in some ways simpler than the above. We consider the linear functional

$$D_\alpha(f)(x) = \lim_{\epsilon\to 0} \frac{1}{c_\alpha} \cdot \int_{|y|\geq\epsilon} \frac{f(x+y)-f(x)}{|y|^{n+\alpha}} \cdot dy, \qquad 0<\alpha<2, \tag{2}$$

where

$$c_\alpha = \pi^{n/2} 2^{-\alpha} \cdot \frac{\Gamma(-\alpha/2)}{\Gamma\left(\frac{n+\alpha}{2}\right)}.$$

We remark that if f is sufficiently restricted (say C_0^∞) then it may be shown that the limit (2) exists (if $0<\alpha<2$), is in L^2, and $D_\alpha(f)^\frown = |x|^\alpha f^\frown(x)$.

THEOREM 2. *Let* $0<\alpha<2$, $1<p<\infty$. *Then* $f\in L^p_\alpha$ *if and only if* (a) $f\in L^p$, *and* (b) *the limit* (2) *defining* $D_\alpha(f)$ *converges in* L^p *norm. Then* $f=J_\alpha(\phi)$ *with*

$$B_{p,\alpha}\|\phi\|_p \leqq \|D_\alpha(f)\|_p + \|f\|_p \leqq A_{p,\alpha}\|\phi\|_p.$$

The proof of this theorem is based on Lemma 4.

It should be pointed out that the integral (2) in general does not converge absolutely; thus its existence depends only in part on the smallness of $f(x+y)-f(x)$. The case $\alpha=1$ is of interest since the integral (2) extends to several variables the "integrated" form of the Hilbert transform.

The following theorem is a straightforward consequence of Theorem 2. It shows that functions in L^p_α can be "localized."

THEOREM 3. *Let* $f\in L^p_\alpha$, $1<p<\infty$, $0\leqq\alpha$, *and* $\psi\in C_0^\infty$. *Then* $\psi\cdot f\in L^p_\alpha$.

The restriction that ψ be indefinitely differentiable can of course be relaxed.

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