RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

A PROOF THAT EVERY BANACH SPACE IS SUBREFLEXIVE

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A real or complex normed space is *subreflexive* if those functionals which attain their supremum on the unit sphere S of E are norm-dense in E^*, i.e., if for each f in E^* and each $\epsilon > 0$ there exist g in E^* and x in S such that $|g(x)| = \|g\|$ and $\|f-g\| < \epsilon$. There exist incomplete normed spaces which are not subreflexive [1][1] as well as incomplete spaces which *are* subreflexive (e.g., a dense subspace of a Hilbert space). It is evident that every reflexive Banach space is subreflexive. The theorem mentioned in the title will be proved for *real* Banach spaces; the result for complex spaces follows from this by considering the spaces over the real field and using the known isometry between complex functionals and the real functionals defined by their real parts.

We first cite a lemma which states, roughly, that if the hyperplanes determined by two functionals f and g (of norm one) are nearly parallel, then one of $\|f-g\|$, $\|f+g\|$ must be small.

LEMMA. *Suppose E is a normed space and $\epsilon > 0$. If $f, g \in E^*$, $\|f\| = 1 = \|g\|$, are such that $|g(x)| \leqq \epsilon/2$ whenever $f(x) = 0$ and $\|x\| \leqq 1$, then $\|f-g\| \leqq \epsilon$ or $\|f+g\| \leqq \epsilon$.*

A proof of the lemma may be found in [2, Lemma 3.1]. To prove the theorem suppose $f \in E^*$ and $\epsilon > 0$. We may assume that $\|f\| = 1$; by the lemma, we want to find g in E^* such that $|g(x)| \leqq 1$ for all x in $T = \{x : f(x) = 0 \text{ and } \|x\| \leqq 2\epsilon^{-1}\}$, and for which there exists x in S such that $g(x) = 1 = \|g\|$. Let C be the convex hull of the union of the sets T and $U = \{x : \|x\| \leqq 1\}$, and *suppose there exists x_0 in U which is also in the boundary of C.* Since C has nonempty interior, by the support theorem there exists g in E^*, $\|g\| = 1$, such that

[1] An easily described example has been suggested by Y. Katznelson: Let E be the space of all polynomials on [0, 1], with the supremum norm. (The example in [1] shows clearly how the method of proof given below fails without the assumption of completeness.)

$\sup\{g(x): x \in C\} = g(x_0)$. It follows that $g(x_0) = 1 = \|x_0\|$, so that the lemma applies and the theorem is proved. Thus, it remains to show that $U \cap \operatorname{bdry} C$ is nonempty. To this end, choose z in U such that $f(z) > 0$ and let $K = (f(z))^{-1}(1+2\epsilon^{-1})$. Define a partial ordering on the set $Z = \{x \in U: f(x) \geqq f(z)\}$ as follows: Say that $x > y$ if

(i) $$f(x) > f(y) \quad \text{and}$$

(ii) $$\|x - y\| \leqq K[f(x) - f(y)].$$

Suppose that W is a totally ordered subset of Z; by (i), the net of real numbers $\{f(x): x \in W\}$ is (bounded and) monotone, and hence converges to its supremum. From (ii) it follows that W is a Cauchy net; by the completeness of E, W converges to a point y in U. By the continuity of f and the norm it follows that y is an upper bound for W. Thus, by Zorn's lemma, there exists a maximal element x_0 of Z; since $x_0 \in U \subset C$, we need only show that $x_0 \in \operatorname{bdry} C$. If not, then x_0 is in the interior of C, and there exists $\alpha > 0$ such that $x_0 + \alpha z \in C$. From the definition of C we see that there exist y in U, x in T and λ in $[0, 1]$ such that $x_0 + \alpha z = \lambda y + (1-\lambda)x$. Then $f(z) \leqq f(x_0) < f(x_0 + \alpha z) = \lambda f(y) \leqq f(y)$, so that $y \in Z$. Also $y - x_0 = (1-\lambda)(y-x) + \alpha z$. Thus, $\|y - x_0\| \leqq (1-\lambda)\|y-x\| + \alpha \leqq (1-\lambda)(\|y\| + \|x\|) + \alpha \leqq (1-\lambda)(1+2\epsilon^{-1}) + \alpha \leqq (1-\lambda+\alpha)(1+2\epsilon^{-1})$. On the other hand, $f(y-x_0) = (1-\lambda)f(y) + \alpha f(z) \geqq (1-\lambda+\alpha)f(z)$, so $\|y - x_0\| \leqq K[f(y) - f(x_0)]$. This shows that $y > x_0$, a contradiction which completes the proof.

A possible generalization of this theorem remains open: Suppose E and F are Banach spaces, and let $\mathcal{L}(E, F)$ be the Banach space of all continuous linear transformations from E into F, with the usual norm. For which E and F are those T such that $\|T\| = \|Tx\|$ (for some x in E, $\|x\| = 1$) dense in $\mathcal{L}(E, F)$? This is true for arbitrary E if F is an ideal in $m(A)$ (the space of bounded functions on the set A, with the supremum norm).

Added in proof: If C is a bounded closed convex set, let $C' = \{f \in E^*: f(x) = \sup\{f(y): y \in C\}$ for some x in $C\}$. A slight modification of the above argument shows that C' is dense in E^*. This solves a problem proposed by Klee in Math. Z. vol. 69 (1958) p. 98.

Bibliography

1. R. R. Phelps, *Subreflexive normed linear spaces*, Arch. Math. vol. 8 (1957) pp. 444–450.

2. ———, *A representation theorem for bounded convex sets*, Proc. Amer. Math. Soc., to appear.

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