A GENERALIZATION OF H-SPACES[1]

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Communicated by Hans Samelson, August 2, 1960

1. **Introduction.** An H-space is a topological space S with a continuous multiplication $f: S\times S\to S$, $f(x, y)=x\cdot y$, *having a two-sided unit* e: thus $e\cdot x=x$, $x\cdot e=x$ for all x in S. We shall consider spaces S with a more general type of product: namely, instead of assuming a two-sided unit e, we assume only

(i) $e\cdot x=x$ for all x.

(ii) There is a continuous map $\sigma: S\to S$ such that $x\cdot\sigma(x)=x$ for all x. Thus if $\sigma(x)=e$ for all x, we have an H-space.

A general class of such spaces S is constructed as follows: let G be a topological group, σ a continuous endomorphism, K a closed subgroup of G *contained in* (not necessarily equal to) the fixed point set of σ; let $S=G/K$, the space of left cosets, and define a product in $S: f(g_1K, g_2K)=g_1\sigma(g_1^{-1})g_2K$. Another way of looking at this product is the following: since G acts on the left on G/K, any continuous map $q: G/K$ into G, defines a product on G/K by $f(g_1K, g_2K)=q(g_1K)g_2K$. In the above situation we have taken the map $q(gK)=g\sigma(g^{-1})$. The product then satisfies (i) and (ii) above, with $\sigma(gK)=\sigma(g)K$. Note that if σ maps all of G onto the identity element, then $S=G$ and the product is just the product in G. We also remark that if q is any *cross-section* of G/K into G (i.e., $\pi q=$ identity map of G/K where $\pi: G\to G/K$, $\pi(g)=gK$) and $q(eK)=e$, the identity element of G, then the multiplication $g_1K\cdot g_2K=q(g_1K)g_2K$ makes G/K an H-space. Such a q is obtained, for instance, if $\sigma^2=\sigma$, $K=\sigma(G)$, and $q=g\sigma(g^{-1})$. We shall be more interested, however, in the case $\sigma^2=I$, the identity map: if, further, K contains the identity component of the fixed point set of σ, then $S=G/K$ is called a symmetric space. The cohomology algebra, with *real* coefficients, of symmetric spaces of compact Lie groups G, is completely known (see [1; 2]); however, with coefficients a field of characteristic $p>0$ less is known and our results when specialized to this case, seem to be new. On taking $G=\mathrm{SO}(n+1)$ the rotation group, $K=\mathrm{SO}(n)$, $G/K=S^n$ and n *odd*, the product in the sphere S^n is essentially the same[2] as one defined by Hopf (in a purely geometric way) in his paper [3] which introduced the subject of H-spaces.

[1] Work supported in part by an N.S.F. grant.

[2] Actually Hopf's product, as is easy to see, is $(g_1K, g_2K)\to g_1\sigma(g_1^{-1})\sigma(g_2)K$, but study of this latter product is equivalent to study of the former.

Our aim is to study the cohomology algebra of the space S using Hopf's method, i.e., using directly the product map f. We assume that the cohomology algebra $H^*(S, F)=\sum_{n\geqq 0} H^n(S, F)$ satisfies three conditions: each H^n is finite-dimensional over F, H^0 is one dimensional, and $H^*(S\times S, F)=H^*(S, F)\otimes H^*(S, F)$. We will also assume F is perfect and contains the eigenvalues of σ^* on $H^*(S, F)$.

2. In this section we assume that we have a space S with a multiplication f satisfying conditions (i) and (ii). We denote by f^* and σ^* the induced cohomology maps, and choose a minimal set of homogeneous generators $z_1, z_2, \cdots$ of $H=H^*(S, F)$ according to the Jordan canonical form of σ^*: i.e. we assume that the degree $d^0z_i\leqq d^0z_{i+1}$, and $\sigma^*(z_i)=\lambda_iz_i+\eta_iz_{i-1}+d_i$, d_i a decomposable element, $\eta_i=0$ or 1. We divide the set of generators into three subsets: $y_1, y_2, \cdots$ being the z_i with $\lambda_i\neq 1$; $x_1, x_2, \cdots$, the z_i with $\lambda_i=1=\eta_i$; $w_1, w_2, \cdots$, the z_i with $\lambda_i=1$, $\eta_i=0$. We denote by Y the subalgebra of H generated by the y_i (and 1).

We call the height of z_i the finite integer s_i (or ∞ if none exists) such that $z_i^{s_i}=0$, $z_i^{s_i-1}\neq 0$. If $z_i=y_j$, we let r_i be the least integer (or ∞) such that $z_i^{r_i}$ belongs to the ideal generated by $z_1, \cdots, z_{i-1}$. Let p = characteristic of F.

PROPOSITION 1. (a) *Let $z_i=y_j$ or x_k. If $p=0$, $s_i=2$ or ∞ (according as d^0z_i is even or odd); if $p>2$, $s_i=2$ if d^0z_i is odd, $s_i\equiv 0$* mod p, *or $s_i=\infty$ if d^0z_i is even; if $p=2$, $s_i\equiv 0$* (mod 2), *or $s_i=\infty$.*

(b) *Let $n_i=0$ if $z_i=w_j$, $n_i=0$ or 1 if $z_i=x_k$, $n_i<r_i$ if $z_i=y_l$. Then $z_1^{n_1}z_2^{n_2}\cdots z_t^{n_t}\neq 0$ for any t. In particular, the subalgebra of H generated by the $z_i=x_j$ or y_k with $z_i^2=0$ is isomorphic to the exterior algebra on these generators.*

(c) *Let $\sigma^2=I$, $p\neq 2$, and choose as generators of H elements $w_1, w_2, \cdots; x_1, x_1', x_2, x_2', \cdots$ satisfying: $\sigma^*(w_i)\equiv w_i$* mod D, *$\sigma^*(x_i)\equiv x_i'$, $\sigma^*(x_i')\equiv x_i$* mod D *(D=decomposable elements). Then: the height of x_i is $\equiv 0$* mod 2, *or $=\infty$; $x_1x_2\cdots x_k\neq 0$ for any k, and $w_ix_jx_{j+1}\cdots x_k\neq 0$ if $d^0w_i\leqq d^0(x_j)\leqq\cdots\leqq d^0(x_k)$.*

THEOREM 1. *Suppose that the subalgebra Y of H generated by the y_i and* 1 *satisfies: $f^*(Y)\subseteq Y\otimes Y$: then H is a free left Y-module. The same result is true with Y replaced by Y', the subalgebra generated by a subset of the y_i. Further, if $f^*(Y')\subseteq Y'\otimes Y'$ then Y' is a Hopf algebra.*

For the next result assume that $f^*\sigma^*=(\sigma^*\otimes\sigma^*)f^*$, σ^* is completely reducible on H, and H is finite-dimensional as a vector space over F. We may then choose the generators as w_i, y_j where $\sigma^*(w_i)=w_i$, $\sigma^*(y_j)=\lambda_jy_j$, $\lambda_j\neq 1$.

Let $y_1, y_2, \cdots, y_r$ be the y_i of lowest degree n.

PROPOSITION 2. *With the above assumptions, if d is the dimension of H as vector space over F, and* $\chi(H) = \sum_i (-1)^i \dim[H^i: F]$, *then*

(a) *If* $p=0$, *then* $d \equiv 0 \mod(2^r)$ *and* $\chi(H)=0$.

(b) *If* $p>0$, *then* $d \equiv 0 \mod(2^r)$ *and* $\chi(H)=0$ *if n is odd, and* $d \equiv 0 \mod(p^r)$, $\chi(H) \equiv 0 \mod p^r$, *if n is even.*

3. We now specialize to the case $S=G/K$. Let π be the projection $\pi(g)=gK$, r the inverse map in G: $r(g)=g^{-1}$, and q: $S \to G$, $q(gK) = g\sigma(g^{-1})$. Then σ^* acts on both $H^*(G)$ and $H^*(G/K)$ (coefficients in the field F). We assume that both $H^*(G)$, $H^*(G/K)$ satisfy the three conditions at the end of 1.

We choose a minimal set of generators $t_1, t_2, \cdots$ of $H^*(G)$ (homogeneous, of increasing degrees) so that $\sigma^*(t_i)=\gamma_i t_i + P(t_1, \cdots, t_{i-1})$, P a polynomial, γ_i in F, and call the generators u_j if $\gamma_i=1$, v_k if $\gamma_i \neq 1$. Similarly we choose generators of $H^*(G/K)$ and call y_i those such that $\sigma^* y_i = \lambda_i y_i + Q_i$, where $\lambda_i \neq 1$ and Q_i involves generators previous to y_i.

Denote by U, V, Y the subalgebras generated by the u_i, v_j, y_k respectively. We assert that the generators can be chosen so that the following is true:

THEOREM 2. (a) $H^*(G)=U \otimes V$, *q^* is an isomorphism of V onto Y, and p^* is* 1-1 *on Y: say* $p^*(Y)=V'$. *Also,* $H^*(G)=U\otimes V'$.

(b) *If, further,* $\sigma^2=I$ *and* $p \neq 2$ *then q^* annihilates the positive degree elements in U, and* $Y=q^*H^*(G)$.

Denote by m: $G \times G \to G$ the map $m(g_1, g_2)=g_1 g_2$

THEOREM 3. *If* $m^*(V) \subseteq V \otimes V$ *and* $\sigma^* r^*(V) \subseteq V$, *then $H^*(G/K)$ is a free left module over* $Y=q^*(V)$. *Similarly, with V replaced by V', a subalgebra generated by a subset of the v_i.*

THEOREM 4. *Let* $\sigma^2=I$, $p\neq 2$. *Then $H^*(G/K)$ is a free left module over* $q^*H^*(G)$.

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