CIRCUMSCRIBED CUBES IN EUCLIDEAN n-SPACE

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Let E^n be a euclidean n-space with a rectangular cartesian coordinate system $(x) = (x_1, \cdots, x_n)$, and let (y) be any system which is a rotation of (x). Let $A \subset E^n$ be a closed bounded set containing $n+1$ linearly independent points. Its circumscribed (y)-box is the set $a_i \leqq y_i \leqq b_i$ $(i = 1, \cdots, n)$ where a_i and b_i are the respective minimum and maximum values of y_i on A. Let $c_i = b_i - a_i$ be interpreted as a function on the space R_{n-1} of rotations of coordinate systems, which is also the rotation space of the unit $(n-1)$-sphere $S^{n-1} \subset E^n$.

Let f: $R_{n-1} \rightarrow E^n$ be the function which maps $r \in R_{n-1}$ onto the point $(c_1(r), \cdots, c_n(r))$, relative to the fixed initial coordinate system (x). Let D be the *diagonal* $x_1 = \cdots = x_n$ in E^n. The circumscribed (y)-box corresponding to a point $r \in R_{n-1}$ is an n-cube if and only if $f(r) \in D$. Accordingly, $K = f^{-1}(D)$, a subspace of R_{n-1}, will be called the *space of circumscribed n-cubes* of A. Its structure can be studied by means of the mapping f. For the purpose of this study the significant properties are as follows: (1) f is a continuous mapping of R_{n-1} into the region $x_i > 0$ $(i = 1, \cdots, n)$ of E^n (2) $f(R_{n-1})$ is symmetric with respect to D. This second property follows from the fact that all possible permutations of axial directions can be achieved in a symmetric way through rotations. There is no need to distinguish between the two possible senses on a given y_i-direction, since the value of c_i is the same for both. Hence, one gets odd as well as even permutations of the c's.

Let T^{n-1} be the simplex in E^n with vertices at the unit points on the (x)-axes. A central projection from the origin carries the mapping f into a continuous mapping g: $R_{n-1} \rightarrow T^{n-1}$ where $g(R_{n-1})$ is symmetric in the barycentric coordinates on T^{n-1}. The inverse image $g^{-1}(q)$, where q is the barycenter of T^{n-1}, is identical with $f^{-1}(D) = K$. This leads to the following result.

THEOREM. *The space of circumscribed cubes of a closed subset of euclidean n-space containing $n+1$ independent points is the inverse image $K = g^{-1}(q)$ of the center of an $(n-1)$-simplex T^{n-1} under a continuous mapping g: $R_{n-1} \rightarrow T^{n-1}$, where R_{n-1} is the rotation space of an $(n-1)$-sphere and where $g(R_{n-1})$ is symmetric in the barycentric coordinates on T^{n-1}.*

Any particular circumscribed n-cube is the (y)-cube for a system (y) obtainable from (x) without rotating any axis by more than

$\pi/2$. A smaller number than $\pi/2$ can be used. Thus all the n-cubes, but not all the rotations yielding them, correspond to points of $K\cap Q^m$ where $Q^m \subset R_{n-1}$ is an m-cell of dimension $m=(n^2-n)/2$, the dimension of R_{n-1}, and where Q^m can be so selected that $g(Q^m)$ is symmetric in the barycentric coordinates on T^{n-1}.

The intersection $K\cap Q^m$ is the inverse image of q under the symmetric mapping $g: Q^m \rightarrow T^{n-1}$ of a closed m-cell into a closed $(n-1)$-cell. This leads to the conclusion that K is of dimension at least $m-(n-1)=(n-1)(n-2)/2$, which is the dimension of the rotation space R_{n-2} of the $(n-2)$-sphere. It is conjectured that K contains a subspace homeomorphic to R_{n-2}, a conjecture easily established for $n=3$. Thus, there exists at least a 1-circuit of circumscribed cubes in the three-dimensional case.

These results strengthen the existence theorems of Kakutani [**1**] in three dimensions and of Yamabe and Yujobô [**2**] in n dimensions.

The above is an outline of material being written up in detail for publication elsewhere.

Bibliography

1. S. Kakutani, *A proof that there exists a circumscribing cube around any bounded closed convex set in R^3*, Ann. of Math. (2) vol. 43 (1942) pp. 739–741.

2. H. Yamabe and Z. Yujobô, *On the continuous function defined on a sphere*, Osaka Math. J. vol. 2 (1950) pp. 19–22.

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