A NEW FORM OF THE GENERALIZED CONTINUUM HYPOTHESIS

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We shall prove that the following condition is equivalent to the generalized continuum hypothesis:

(*) *For all transfinite cardinals* p *and* q, *if* p *covers* q, *then for some* r, $p=2^r$.

By p covers q, we mean that $p>q$ and for no r is $p>r>q$.

The generalized continuum hypothesis is usually stated in the form that, for any transfinite cardinal p, 2^p covers p. We shall use instead the equivalent form [2; 4] as the logical product of the aleph hypothesis $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ and the axiom of choice.

If the generalized continuum hypothesis holds, then (*) follows easily. For then if p and q are transfinite and p covers q, then by the axiom of choice for some α, $q=\aleph_\alpha$ and $p=\aleph_{\alpha+1}$ and so by hypothesis $p=2^q$.

Let us now proceed to the converse. First we shall prove the aleph hypothesis. Since for all α, $\aleph_{\alpha+1}$ covers $\aleph_\alpha$, we have $\aleph_{\alpha+1}=2^r$ for some r. Since $r<2^r$, r must be $\aleph_\gamma$ for some γ. Let $\beta(\alpha)$ be the smallest such γ. We clearly have $\beta(\alpha)<\alpha+1$. However, $\beta(\alpha)$ is a strictly monotone function of α and hence is greater than or equal to α. Thus $\beta(\alpha)=\alpha$ and the aleph hypothesis is proved.

Let us now demonstrate that the axiom of choice follows from (*). We first prove from the axioms of set theory the following

LEMMA.[1] *If* $2^p\leqq q+\aleph_\alpha$, *where* p *and* q *are transfinite, then* $p<q$ *or* $p<\aleph_\alpha$.

For since $p<2^p$, $p=s+t$, where $s\leqq q$ and $t\leqq\aleph_\alpha$. Then $2^p=2^s2^t$, and by [2] either $2^s\leqq\aleph_\alpha$ or $2^t\leqq q$. But in the first case $s+t\leqq\aleph_\alpha$ since both s and t are, and in the second case $s+t\leqq q$ since both s and t are, and in addition t is less than or equal to an aleph. Thus we have demonstrated the lemma except for the strictness of the inequalities. That follows since [2; 5] if $2^p\leqq p+r$, then $2^p\leqq r$, and $p<r$, q.e.d.

For any transfinite cardinal p, let us denote by p^* the smallest aleph [1] not less than or equal to p. Tarski [3] has shown that if p is transfinite then $p+p^*$ covers p. But since by [2] the mapping $p\rightarrow p^*$

[1] This lemma is due to Professor A. Tarski and is an extension of the author's original argument.

preserves addition and since $\aleph_\alpha^* = \aleph_{\alpha+1}$, it follows that if $\aleph_{\alpha+1}$ is not less than or equal to p then $p+\aleph_{\alpha+1}$ covers $p+\aleph_\alpha$. Then by (*), $p+\aleph_{\alpha+1}=2^q$, and we have $q<p$ or $q<\aleph_{\alpha+1}$. However, if we choose $\aleph_{\alpha+1} \geqq (2^p)^*$, the first alternative is impossible, and q is an aleph. Then by the aleph hypothesis, 2^q is an aleph, and so p is an aleph.

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