NORMAL OPERATORS ON THE BANACH SPACE $L^p(-\infty, \infty)$ PART I. BOUNDED OPERATORS[1]

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1. **Introduction.** Let $\mathcal{B}$ denote the Boolean ring generated by the semi-closed intervals of the plane. We denote by $\mathfrak{E}_p$ the algebra of all linear bounded transformations of $L^p(-\infty, \infty)$ into itself. Suppose for a moment that $p=2$, and let $\mathfrak{A}_p$ be an involutive abelian subalgebra of $\mathfrak{E}_p$: if $\mathfrak{A}_p$ is also a Banach space and if $T_p \in \mathfrak{A}_p$, then

(i) *the family of all homomorphic mappings of the ring* $\mathcal{B}$ *into the algebra* $\mathfrak{E}_p$ *contains a member* E_p^T *such that*

$$T_p = \int \lambda \cdot E_p^T(d\lambda). \tag{1}$$

Suppose henceforth that $1 < p < \infty$. To which extent does the preceding situation carry over?

Let $\mathfrak{D}$ be the class of all bounded functions whose real and imaginary parts are piecewise monotone. In §2 will be defined an isomorphism $f \to \wedge(f)_p$ whose domain includes $\mathfrak{D}$ and whose range $(t)_p$ is a normed[2] involutive abelian subalgebra of $\mathfrak{E}_p$. Our main theorem (§3) shows that a member T_p of $(t)_p$ has the property (i) whenever $T_p = \wedge(f)_p$ for some f in $\mathfrak{D}$. The relation (1) involves a Stieltjes integral defined in the strong operator-topology whenever $p \geqq 2$. The set-function E_p^T need not be countably additive: we do not restrict ourselves to "*spectral resolutions*" in the sense of Dunford (however, condition (ii) in [**1**, p. 219] is satisfied if B and E are replaced by our $\mathcal{B}$ and E_p^T). The values of E_p^T are self-adjoint [**2**, p. 22] idempotent members of $(t)_p$.

It is easily seen that the Hilbert transformation and the Dirichlet operators all have the property (i). For less trivial examples, let $\mathfrak{M}^1$ be the set of all bounded Radon measures on $(-\infty, \infty)$; if $A \in \mathfrak{M}^1$, then the convolution operator A_{*p} is defined as the mapping $x \to A * x$ of $L^p(-\infty, \infty)$ into itself. In case the Fourier transform of A belongs to $\mathfrak{D}$, then the operator $T_p = A_{*p}$ has the property (i). Consequently,

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[2] In the terminology of Neumark [**5**, p. 110]: the normed ring $(t)_p$ has an isometric involution, and its completion is a (*)-algebra [**2**, p. 22].

all the classical convolution operators (Weierstrass, Picard, Poisson, Stieltjes, Fejér, etc.) have the property (i).

The differentiation operator D_p is in a class of unbounded operators having the property (i). Although details regarding such unbounded operators will be reserved for Part II of this announcement, it may be pertinent to remark here that a relation of the type

$$T_p = \int_{-\infty}^{\infty} \cdot f(\theta) \cdot \overset{D}{E_p}(d\theta)$$

holds for any T_p such that $T_p = \wedge (f)_p$ for some function f of locally bounded variation. For example, take α in $(-\infty, \infty)$ and let T_p be the translator defined by $T_p x(\theta) = x(\theta - \alpha)$ for all x in $L^p(-\infty, \infty)$; then

$$T_p = \int_{-\infty}^{\infty} e^{2\pi i\alpha\theta} \cdot \overset{D}{E_p}(d\theta).$$

2. **The basic algebra.** Denote by L^+ the intersection of the family $\{L^p(-\infty, \infty): 1 < p < \infty\}$, set $\mathfrak{F} = L^\infty(-\infty, \infty)$, and let $\mathfrak{E}$ be the set of all linear operators T which map L^+ into itself and which satisfy the condition $\|T\|_p \neq \infty$ whenever $1 < p < \infty$.

If $f \in \mathfrak{F}$, then $t(f)$ is defined as the set of all T in $\mathfrak{E}$ such that

$$(\text{Fourier transform of } Tx) = (\text{Fourier transform of } x) \cdot f$$

for all x in L^+. It is readily checked that the union (t) of the family $\{t(f): f \in \mathfrak{F}\}$ forms a subalgebra of $\mathfrak{E}$. If $T \in (t)$, let $\vee T$ denote the "unique" bounded function f in $\mathfrak{F}$ such that $T \in t(f)$ (equivalent functions being identified). The mapping $T \rightarrow \vee T$ maps (t) isomorphically onto a subalgebra $\mathfrak{F}_v$ of $\mathfrak{F}$. The set $\mathfrak{F}_v$ contains all functions of bounded variation on $(-\infty, \infty)$. If $g \in \mathfrak{F}_v$, let $\wedge (g)$ denote the inverse image of g under the mapping $T \rightarrow \vee T$. For example, $-i \cdot \operatorname{sgn} \in \mathfrak{F}_v$ and $\wedge(-i \cdot \operatorname{sgn})$ is the Hilbert transformation. If $T \in t(g)$ then $\overline{T} = \wedge (\bar{g})$ by definition (where $\bar{g}$ = complex conjugate of g). If

$$(x \mid y) = \int_{-\infty}^{\infty} x(\theta) \cdot \bar{y}(\theta) \cdot d\theta$$

then $(Tx \mid y) = (x \mid \overline{T}y)$ whenever both x and y are in L^+; moreover $\|T\|_p = \|\overline{T}\|_p$.

From the definition of $\mathfrak{E}$ follows that any T in $\mathfrak{E}$ has a unique extension T_p in $\mathfrak{E}_p$. The involutive algebra $(t)_p$ is as the isomorphic image of (t) under the mapping $T \rightarrow T_p$.

All convolution operators A_{*p} belong to $(t)_p$ (when $A \in \mathfrak{M}^1$). A fundamental role is played by the Dirichlet operator $\wedge(\phi_\alpha)$, where ϕ_α is the characteristic function of the interval $(-\alpha, \alpha)$.

3. **The main result.** The isomorphism between $\mathfrak{F}_v$ and $(t)_p$ permits the application of the machinery set up in Part I of [**4**]. The outcome is the following

THEOREM. *If $T_p \in (t)_p$ and if $\vee T$ belongs to $\mathfrak{D}$, then T_p has property* (i), *and the relation* (1) *holds in the strong operator-topology when $p \geqq 2$; it holds in the weak topology when $1 \leqq p \leqq \infty$.*

In conclusion, a few remarks about the relation (1). Set $g = \vee T$ and denote by $g☆(\mathfrak{a})$ the characteristic function of the inverse image (under g) of the set $\mathfrak{a}$; the set-function E^T is defined for all $\mathfrak{a}$ in $\mathfrak{B}$ by the relation $E^T(\mathfrak{a}) = \wedge(g☆(\mathfrak{a}))$. Consider the set $\mathfrak{N}_g$ of all marked partitions $\Pi = (r, R)$ of the range of g; the set $\mathfrak{N}_g$ is directed by the usual Riemann ordering $\gg$ (decreasing meshes: see [**4**] or [**3**, p. 79]). The integral in (1) is to be interpreted in the Riemann-Stieltjes sense: if $p \geqq 2$ then (1) states that the net

$$\left\{ \left\| T_p x - \sum_\nu r_\nu (E^T(R_\nu))_p x \right\|_p ; \ \Pi \in (\mathfrak{N}_g, \gg) \right\}$$

converges to 0 (Moore-Smith convergence) for all x in $L^p(-\infty, \infty)$.

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