

be considered too hard (pp. 17, 207, 305) and some of the applications too specialized (pp. 265–271) or too controversial (pp. 291–305) for inclusion in an elementary text. One of the exercises (ex. 1 on p. 194) is likely to lead to confusion without some fairly detailed discussion on the part of the instructor. (E.g. does “All predicted responses were reinforced” mean, “It was predicted that all responses would be reinforced,” or, “All those responses which were (correctly) predicted were (predicted and observed to be) reinforced”?) Most instructors will find it desirable to add some mathematical illustrations to the almost exclusively verbal ones in Chapters I–VI. One or two choices of subject-matter are perhaps slightly questionable; for example I would rather see an elementary treatment of homomorphism around p. 220 than the curious discussion of the computation of the inverse of a (real) function which occupies pp. 235–240, and I would rather see Chapter VI on use and mention curtailed and embodied in the chapter on functions (which is what most instructors will do with it anyway). Finally the quantificational rules of Chapter IV can be somewhat simplified with no loss in rigour (e.g. along the lines of Fitch’s *Symbolic logic*, §§ 21.12–13 and 22.8–9).

But all these are very minor quibbles reflecting as much as anything biographical idiosyncrasies and accidental encounters with students. Try as I will, I cannot find anything serious to complain about. The book comes as near to a perfect fulfillment of its function in the rough-and-tumble of the classroom as any you are likely to find. Clearly it is destined to become a classic and not be soon replaced. I can only hope that it will stimulate educators to try the effect of an early rigorous logical training, perhaps compulsory, on science majors generally. It was a thankless task indeed to make this experiment before the publication of Suppes’ book; now it has become a challenge and an adventure.

NOTE. The first printing (1957) was marred by a large number of printer’s errors. However of those which caught the attention of this reviewer, only two remain in the second printing (1958). P. 167, l. 7, replace “first” by “second”; p. 172, l. 7 from bottom, replace “independent of” by “dependent on.”

JOHN MYHILL

Asymptotic methods in analysis. By N. G. de Bruijn. Amsterdam, North-Holland; Groningen, Noordhoff; New York, Interscience, 1958. 12+200 pp. \$5.75.

This book is for you if you are interested in answering questions like the following: What is a good approximate formula for x if

$xe^x = t$? How does $\sum_{n=1}^{\infty} n^{-1}e^{-n^2t}$ behave as $t \rightarrow 0$? In how many ways, approximately, can you partition a set containing a large number n of elements into disjoint nonempty subsets? What is the behavior of

$$(*) \quad \sum_{k=0}^{2n} (-1)^{k+n} \binom{2n}{k}^s$$

for large n ? What happens to the n th iterate of $\sin x$ when $n \rightarrow \infty$? How do the solutions of $y''(t) - t^4y(t) = 0$ or of $t^{-k}y'(t) = \alpha(t) + \beta(t)y(t) + \gamma(t)y^2(t)$ behave as $t \rightarrow \infty$?

This is a textbook, not a systematic treatise, and it a textbook of an unusual kind. It contains no general theory, and the author refrains from formulating general theorems. As he rightly says, any attempt to do so would result in a loss of generality, since any theorem is inapplicable to some special case. The emphasis is on methods and their illustration by examples, preferably nontrivial ones. The whole book is written in a pleasantly informal style with many attempts to indicate the motivation for what is done. Here we have an expert letting us in on the tricks of his trade, instead of building an impressive and logically organized structure. Even so, this is not an easy book, but then the subject is inherently difficult. It demands great technical competence (which is not fashionable at the moment); but there are parts of mathematics and physics where it is nevertheless useful. Concerning its relevance for actual numerical work, the author has this to say:

"But even if the asymptotic result is presented in its best possible explicit form, it need not be satisfactory from the numerical point of view. The following dialogue between Miss N.A., a Numerical Analyst, and Dr A.A., an Asymptotic Analyst, is typical in several respects.

N.A.: I want to evaluate my function $f(x)$ for large values of x , with a relative error of at most 1%.

A.A.: $f(x) = x^{-1} + O(x^{-2})$ ($x \rightarrow \infty$).

N.A.: I am sorry, but I don't understand.

A.A.: $|f(x) - x^{-1}| < 8x^{-2}$ ($x > 10^4$).

N.A.: But my value of x is only 100.

A.A.: Why did not you say so? My evaluations give

$$|f(x) - x^{-1}| < 57000x^{-2} \quad (x \geq 100).$$

N.A.: This is no news to me. I know already that $0 < f(100) < 1$.

A.A.: I can gain a little on some of my estimates. Now I find that

$$|f(x) - x^{-1}| < 20x^{-2} \quad (x \geq 100).$$

N.A.: I asked for 1%, not for 20%.

A.A.: It is almost the best thing I possibly can get. Why don't you take larger values of x ?

N.A.: !!! I think it's better to ask my electronic computing machine.

Machine: $f(100) = 0.01137\ 42259\ 34008\ 67153$.

A.A.: Haven't I told you so? My estimate of 20% was not very far from the 14% of the real error.

N.A.: !!! . . . !.

Some days later, Miss N.A. wants to know the value of $f(1000)$. She now asks her machine first, and notices it will require a month, working at top speed. Therefore, she returns to her Asymptotic Colleague, and gets a fully satisfactory reply."

The book begins with a general introduction on O and o notation, and asymptotic series in general. After this, the various topics are pretty much independent of each other: estimation of implicit functions and roots of equations; various methods for estimating sums; three chapters on the saddle-point method, with full details and quite intricate worked examples, such as the asymptotic behavior of $\int_0^\infty e^{-P(u)} u^{s-1} du$, where P is a polynomial with complex coefficients; a brief introduction to Tauberian theorems; a detailed chapter on iteration; and a fairly short chapter on asymptotic behavior of solutions of differential equations. There is, as far as I know, no other book with such an elaborate account of the saddle-point method; and many of the illustrative results throughout the book will not be found anywhere else.

A couple of special results deserve to be mentioned because they show asymptotic techniques of the most sophisticated kind furnishing information about problems that seem not at all asymptotic to begin with. The sum $S(s, n)$ (formula (*) above) is known for

$$s=2, 3: S(2, n) = (2n)!/(n!)^2 \text{ and } S(3, n) = (3n)!/(n!)^3.$$

The asymptotic formula for $S(s, n)$ turns out to involve $(\cos \pi/2s)^{2ns}$, which is not rational except for $s=2$ or 3 , so that the asymptotic discussion (which the author gives for all real s) indicates that no simple formula for $S(s, n)$ can exist for $s > 3$.

If $a_n \geq 0$ Copson's inequality

$$\sum_{n=1}^{\infty} a_n \leq \gamma \sum_{n=1}^{\infty} n^{-1/2} (a_n^2 + a_{n+1}^2 + \dots)$$

(cf. Hardy, Littlewood, and Pólya, *Inequalities*, Cambridge, 1934, Theorem 345) holds with $\gamma = 2^{1/2}$; the author shows that the deter-

mination of the best (i.e., smallest) γ depends on the solution of the system $u_1 = x$, $u_2 = 2^{-1/2}x + (u_1^2 - 1)^{1/2}$, $u_3 = 3^{-1/2}x + (u_2^2 - 1)^{1/2}$, \dots . A detailed study of the asymptotic behavior of u_n , followed by numerical computations, showed that $\gamma = 1.10649577 \dots$.

The author refers to Erdélyi, *Asymptotic expansions*, New York, Dover, 1956, for bibliography and more material on differential equations. Reference could also be made to the Russian monograph by Evgrafov (*Asymptotic estimates and entire functions*, Moscow, 1957), which contains many techniques and results for problems of a rather special kind.

R. P. BOAS, JR.

Multivalent functions. By W. K. Hayman. Cambridge Tract, no. 48. New York, Cambridge University Press, 1958. 8+151 pp. \$4.00.

To the reviewer's knowledge there is no other topic of mathematics of comparable depth and sophistication to the theory of univalent functions in which it is possible to approach the same results by so many quite distinct methods (at least in the formal sense). To present a complete account of all aspects of the theory of univalent functions, to say nothing of its extensions to multivalent functions, would require an exceptionally large treatise. It is thus to be expected that any presentation of modest size will constitute a treatment of one or more particular aspects of the subject. In the present case the author quite naturally deals principally with that direction of the theory which is closest to his own work in the subject. This may be characterized as the study of univalent and multivalent functions by the traditional methods of analysis. This is not to say that the author confines himself to elementary methods (which serve to prove only the simplest results). Indeed he makes use in the proof of his principal objectives of the method of the extremal metric (at least in its primitive form of the length-area principle) and the method of symmetrization. However these methods are used to derive certain individual lemmas and theorems and the main stream of the argument follows the classical inequality proof pattern.

The present tract consists of six chapters. Of these the first and the last are somewhat apart from the main portion of the book. The first chapter is essentially a very brief survey of the most elementary parts of the theory of univalent functions. No further comment seems called for except to point out the one result here due to the author (Theorem 1.4) which plays an important role in later developments. Chapter six is an exposition of Löwner's parametric method and some of its applications. It depends on the intervening chapters