SOME PROBABILITY LIMIT THEOREMS

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We are concerned with the partial sums $S_0=0$, $S_n=X_1+\cdots+X_n$ of identically distributed independent random variables X_i with mean zero and finite positive variance, i.e.

$$(1) \qquad E(X_i) = 0, \qquad 0 < \sigma^2(X_i) = \sigma^2 < \infty.$$

Some of the present results describe new phenomena, whereas others are refinements of known theorems. While they deal with the limiting behavior of certain functionals of the partial sums, they appear to be of a type which cannot be reduced to the study of a functional of the Wiener process. With the exception of Theorem 6, nothing but (1) will be assumed about the common distribution of the X_i.

We define the probabilities:

$$
\begin{aligned}
& & c_k &= \Pr[S_k > 0], & k &\geqq 1;\\
p_0 &= 1, & p_n &= \Pr[S_1 > 0, \cdots, S_n > 0], & n &\geqq 1;\\
q_0 &= 1, & q_n &= \Pr[S_1 \leqq 0, \cdots, S_n \leqq 0], & n &\geqq 1;
\end{aligned}
$$

and the random variables (which by virtue of (1) exist and are finite with probability 1):

Z = the first positive term in the infinite sequence $S_1, S_2, \cdots$;

N_n = the number of positive terms in the finite sequence $S_1, S_2, \cdots, S_n$;

$N_A(I)$ = the number of terms S_k in the infinite sequence $S_1, S_2, \cdots$, such that $S_k \in I$ and $S_i \leqq A$ for $i=1, 2, \cdots, k$;

$N_A^*(I)$ = the number of terms S_k in the infinite sequence $S_1, S_2, \cdots$, such that $S_k \in I$ and $|S_i| \leqq A$ for $i=1, 2, \cdots, k$.

Here A is a positive number and I is a closed bounded interval. $\mu(I)$ will denote the length of I when the X_i are nonlattice random variables, and the number of integers in I when the X_i are lattice random variables such that the smallest group containing all possible values of all the partial sums S_n is the group of all integers.

THEOREM 1. *The series* $\sum_1^\infty k^{-1}(1/2-a_k)$ *converges* (*conditionally*; *probably not always absolutely*).

The constant $c=\exp \sum_1^\infty k^{-1}(1/2-a_k)$ plays an important role in

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what follows. Note that always $0<c<\infty$, while $c=1$ for symmetric random variables X_i with a continuous distribution.

THEOREM 2.

$$E(Z) = c\cdot\sigma/2^{1/2}.$$

THEOREM 3.

$$\lim_{n\to\infty} (n\pi)^{1/2}p_n = \frac{1}{c}, \qquad \lim_{n\to\infty} (n\pi)^{1/2}q_n = c.$$

The methods of proof of these results are refinements of those in [**1**; **4**; **5**]. The moment generating function of Z was first discovered by Baxter. In the simplest case, when the X_i are symmetric with a continuous c.d.f. the proof of Theorem 2 follows from Karamata's Tauberian theorem applied to Baxter's identity

$$(2) \qquad E[e^{-\lambda Z}] = 1 - \exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\lambda}{\lambda^2+u^2}\log[1-\phi(u)]du\right\},$$
$$\lambda > 0, \qquad \phi(u) = E[e^{iuX_k}].$$

THEOREM 4.

$$(k(n-k))^{1/2}\Pr[N_n = k] = 1/\pi + o(1),$$

where $o(1)$ *is a function of* k *and* n *such that* $o(1)\to 0$ *as* $\min(k, n-k)\to\infty$.

This is an immediate corollary of Theorem 3 combined with Sparre Andersen's theorem [**1**]

$$\Pr[N_n = k] = p_k q_{n-k}.$$

It implies, but is stronger than, the celebrated arc-sine law for identically distributed random variables [**1**; **2**; **4**].

THEOREM 5.

$$\lim_{n\to\infty} E\left[\frac{S_n}{(n\sigma^2)^{1/2}} \mid S_1 \geqq 0, S_2 \geqq 0, \cdots, S_n \geqq 0\right] = \left(\frac{\pi}{2}\right)^{1/2}.$$

This result follows easily from Theorems 2 and 3. It gains interest by comparison to

$$\lim_{n\to\infty} E\left[\frac{S_n}{(n\sigma^2)^{1/2}} \mid S_n \geqq 0\right] = \left(\frac{2}{\pi}\right)^{1/2}.$$

THEOREM 6. *The X_i are assumed to be symmetric a-periodic lattice with finite positive variance, i.e. random variables*

$$c_k = \Pr[X_i = k] = \Pr[X_i = -k] = c_{-k}, \quad 0 < \sigma^2 = \sum_{-\infty}^{\infty} k^2 c_k < \infty,$$
$$\text{g.c.d.}[k \mid c_k > 0, k \geqq 1] = 1.$$

The interval I is assumed to be a single, fixed integer, so that $\mu(I) = 1$. Then

(a) $$\lim_{A\to\infty} \Pr\left[N_A(I) \leqq \frac{2A}{\sigma^2}\mu(I)x\right] = \begin{cases} 1 - e^{-x}, & x \geqq 0, \\ 0, & x < 0. \end{cases}$$

(b) $$\lim_{A\to\infty} \Pr\left[N_A^*(I) \leqq \frac{A}{\sigma^2}\mu(I)x\right] = \begin{cases} 1 - e^{-x}, & x \geqq 0, \\ 0 & x < 0. \end{cases}$$

There is a close connection between this result and the considerations behind Theorem 2. Both are expressions of the asymptotic behavior of the resolvent of the substochastic transition operator of the stochastic process $\{S_n\}$ in the presence of an absorbing barrier (at $+A$ in Part (a) and at $\pm A$ in Part (b)). The proof of Part (b) was found in collaboration with C. Stone [7], using methods from the theory of Toeplitz forms [3], where geometric means such as the last term in Equation (2) play a crucial role.

The intuitive content of Theorem 6 as an occupation time theorem makes it extremely plausible that it holds for arbitrary X_i satisfying (1), integer-valued and aperiodic if they are of the lattice type, with I and $\mu(I)$ given in the definition of the occupation times $N_A(I)$ and $N_A^*(I)$.

The proofs of Theorems 1 through 5 will appear in [6] and that of Theorem 6(b) in [7].

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