The material has been carefully planned and developed in a clear and simple style, and the proofs are complete, neat and compact.

The introduction of the Lebesgue measure is based on the method of M. Riesz (Ann. Soc. Pol. Math. vol. 25 (1952)). The authors' aim is to present the theory in a form suitable for a person having the knowledge of elementary calculus without however weakening the theorems. This is excellently carried out by the clear exposition, explanations and a thorough treatment of the chosen material. There are no problems in the book.

In the opinion of the reviewer this book is an exceptionally good one and most suitable as a textbook for courses in which the concepts of the Lebesgue measure and integral are essential.

STANISLAW LEJA

Kontinuierliche Geometrien. By Fumitomo Maeda. Trans. from the Japanese by Sibylla Crampe, Gunter Pickert and Rudolf Schauffler. Springer-Verlag, Berlin, 1958. 10+244 pp. DM 36. Bound DM 39.

Continuous geometries are a generalization of the finite dimensional projective geometries to the non-finite dimensional case, as Hilbert and Banach spaces are a generalization of the finite dimensional (Minkowski) vector spaces.

It is now 23 years since John von Neumann first discovered continuous geometry. As a result of his work on rings of operators in Hilbert space (partly in collaboration with F. J. Murray), von Neumann found that certain families of closed linear subspaces of Hilbert space had intersection (i.e., incidence) properties very much like the intersection properties possessed by the set of all linear subspaces of a finite dimensional projective geometry. Profiting by the previous work of Dedekind, G. Birkhoff, Ore and Menger in the field now called lattice theory, von Neumann gave a set of axioms to describe such families of subspaces of Hilbert space (and certain abstractions of these, which he called continuous geometries) as complete lattices in which every element possesses at least one complement, and which satisfy a weak distributivity condition (first formulated by Dedekind, and called now the modular axiom). Von Neumann, at first, required the lattice to be also irreducible and to satisfy certain continuity conditions on the lattice operations.

Von Neumann's first deep result was the construction of a dimension function $D(a)$, defined for each element a in the geometry, with $0 \leqq D(a) \leqq 1$ for all a, and satisfying the usual condition: $D(a \cup b) + D(a \cap b) = D(a) + D(b)$. For this purpose von Neumann assumed both irreducibility and the continuity conditions. He formulated the lattice-abstraction of the notion: for two elements in the geometry to be perspective, and used the relation "to be perspective" as a method of defining equidimensionality. By a new and delicate analysis, he showed that the technique he had previously used, in his papers on Haar measure and dimension theory for rings of operators, could be now applied to the case of abstract continuous geometries, and yielded the desired dimension function.

While lecturing at Princeton during 1936–1937 von Neumann made a whole series of brilliant discoveries in connection with his new geometries. He dropped the irreducibility axiom (which meant that the centre of the lattice would be non-trivial) and gave a powerful analysis of the centre in this case. This permitted him to construct a dimension function even in the non-irreducible case but now the dimension function was vector-valued.

The next result, undoubtedly the deepest and the most exciting of all, was his coordinatization theorem. In the theory of projective geometry, the work of many mathematicians, including von Staudt, Hessenberg, Hilbert, Veblen and Young, had produced the now classical result that every projective geometry satisfying Desargues' Theorem (in particular, having dimension more than two) could be coordinatized, using homogeneous coordinates for the points of the geometry. Von Neumann expressed this classical theorem in two slightly different forms (each of which was more suitable for the generalization he was about to establish) as follows: let $\mathfrak{R}$ denote a division ring, let $\mathfrak{R}^{(n)}$ denote the right module of all vectors $x = (x_1, \cdots, x_n)$ with all x_i in $\mathfrak{R}$, and let $\mathfrak{R}_n$ denote the ring of all matrices $x = (x_{ij}; i, j = 1, \cdots, n)$; then, for each $n-1$ dimensional projective geometry (satisfying Desargues' Theorem), there exists a suitable division ring $\mathfrak{R}$ such that the class of all the linear subspaces of the given geometry, partially ordered by inclusion, can be put in lattice isomorphism with the class of all the right submodules of $\mathfrak{R}^{(n)}$, and can also be put in lattice isomorphism with the class of all the right ideals of R_n. Then von Neumann gave a point-free, purely lattice theoretic discussion, which applies to any complemented modular lattice (of dimension not too low) and which generalizes the projective geometry coordinatization theorem in a remarkable way. The division ring had now to be replaced by a more general type of ring, which von Neumann defined and which he called a regular ring; then, in the statement of the theorem, right submodule had to be replaced by right submodule of finite span and right ideal had to be replaced by principal right ideal. The notion of dimensionality was replaced by the notion of a homogeneous basis of order n and the theorem was established under the assumption that the order n exceeds three (this, for projective geometry, means that the dimension exceeds two).

These two results, i.e., the dimension theory and the coordinatization theory, were written up in lecture notes distributed at Princeton in 1935–1937. Though the notes were by no means in the final form that von Neumann intended, yet they were detailed and, except for a few rather minor slips, rigorous (the discussion of the dimension theory for the non-irreducible case is not completed in the Princeton notes, which break off almost in the middle of a statement).

After lecturing before the American Mathematical Society in 1937 on these and other results, and after publishing five articles in the Proceedings of the National Academy giving abstracts of his work, von Neumann proposed to write a book on Continuous Geometry to appear in the Colloquium series of the American Mathematical Society. But his work in the theory of games, other interests, and the war, intervened. As the years went by, he finally decided that the Princeton notes, at least, should be reproduced by the Princeton University Press. Even this did not happen before his death.

The book under review first appeared in Japanese in 1950 or 1951. It now appears, with a few improvements but apparently no drastic changes, in German. The translators, S. Crampe, Gunter Pickert and Rudolf Schauffler, have done their work in superb fashion; in view of the unusually involved mathematical theory along with the linguistic difficulties, they may well be proud of their achievement. The printing is clear, almost completely free of error, and the notation and general style are excellent. As for the material presented by the author, this is dominated by those parts which are an exposition of the notes of von Neumann (with minor changes from time to time).

The book is divided into 12 chapters and two appendices, as follows: Chapter One is devoted to lattice theory in general, Chapter Two to modular lattices, Chapter Three to projective spaces, Chapter Four to certain fundamental properties of continuous complemented modular lattices, and Chapter Five to the dimension theory for such lattices. Some repetition in the von Neumann notes is avoided, some shortening of the proof is obtained by repeating an argument of von Neumann and the reviewer, and the completion of the dimension theory (which did not get included in von Neumann's notes, as mentioned above) follows an article of T. Iwamura. A more general dimension theory due to the reviewer (associated still with perspectivity) is mentioned in a footnote but no mention is made of the second paper by T. Iwamura which began an abstraction of the dimension theory (later expanded in the work of L. Loomis and S. Maeda, son of the author, to more general lattices).

The next six chapters are devoted to regular rings and the coordinatization theorem. Chapter Six gives various equivalent ways of defining a regular ring (e.g., for each x in the ring there is an element y in the ring such that $xyx=x$). The author follows von Neumann's notes closely, requires the ring to possess a unit element and gives von Neumann's proof that if $\mathfrak{R}$ is a regular ring with unit then its principal right ideals under inclusion form a complemented modular lattice (more recent articles, for instance by K. D. Fryer and the reviewer, show that in every regular ring the principal right ideals form a relatively complemented modular lattice). Chapter Seven is devoted to continuous regular rings and their rank functions; this chapter also presents material mostly from von Neumann's notes but adds some detail on the explicit representation of reducible regular rings as subdirect products of irreducible ones. This chapter is not required for the coordinatization theorem, which is covered in Chapters Eight, Nine, Ten and Eleven. There, some important variations from the notes of von Neumann are attributed to Kodaira and Furuya, who published three articles on Continuous Geometry in 1938, in Japanese (apparently unavailable now). Kodaira and Furuya wrote in such a way as to avoid one of two slips in the notes of von Neumann (both slips are corrected in the book under review). These slips were pointed out explicitly much later by H. Löwig to von Neumann, who acknowledged the slips but commented that in writing the notes he had chosen hurriedly from various alternative proofs which he had found, so that these slips could be easily remedied by him. In fact, a very long, involved and invalid proof of Theorem 13.1 in Part II of the notes (replaced by a shorter three page proof by Kodaira and Furuya, Satz 3.7 in the book under review) can be included in a much more general theorem which can be proved in a few lines, as follows: Theorem (found by von Neumann and the reviewer in 1937). Let x be an element in a relatively complemented modular lattice and suppose $a \leqq b$; then there exists a relative complement y of a in b such that $x=(x \cup y) \cap (x \cup a)$. To prove this, let $x_1 = x \cap a$; let x_2 be a relative complement of $x \cap a$ in $x \cap b$; let x_3 be a relative complement of $x \cap b$ in x; let y_1 be a relative complement of $a \cap x$ in a; let y_2 be a relative complement of $a \cup x_2$ in b. Then x_1, x_2, x_3, y_1, y_2 are independent and it suffices to choose $y = x_2 \cup y_2$.

The last chapter discusses complemented modular lattices which possess an orthocomplementation and regular rings which possess an involutoric anti-automorphism and the relation between these two systems. Along with the theory given by von Neumann, the author gives his generalization of the theorem of G. Birkhoff and von Neumann, proved by those authors for the finite dimensional (projective) geometries.

The first appendix shows the equivalence of the axiom of choice, the well-ordering theorem, and Zorn's Lemma; the second appendix gives various ways (all equivalent) to define continuity of the lattice operations in a complemented modular lattice (some of these ways are more convenient for repeated use in proofs than others).

There is a footnote reference to the remarkable discovery of I. Kaplansky that every complete complemented modular lattice which is orthocomplemented is necessarily continuous but there is no other reference to work done in the field of continuous geometry after 1951.

ISRAEL HALPERIN

Théorie des ensembles (Chapter III). By N. Bourbaki. Actualités Scientifiques et Industrielles, no. 1243, Paris, Hermann, 1956. 118 pp. 1500 fr.

This third chapter of Bourbaki's set theory is entitled *Ensembles ordonnés. Cardinaux. Nombres entiers.* The text deals in the Bourbaki fashion with those elementary parts of these subjects that are needed in the later books of the Bourbaki treatise, and is divided into six sections: 1. Order relations, ordered sets; 2. Well-ordered sets (including transfinite induction and the well-ordering theorem); 3. Equivalent sets, cardinals (including Cantor's theorem that $2^{\mathfrak{a}} > \mathfrak{a}$); 4. Finite cardinals, finite sets (including mathematical induction); 5. Operations with integers (including combinatorial analysis); 6. Infinite sets (including the theorem that $\mathfrak{a}^2 = \mathfrak{a}$ if $\mathfrak{a}$ is infinite).

About 35 of the 118 pages of the book consist of exercises, to which are relegated such important notions as order types, ordinal numbers, alephs, initial as well as regular, singular, indecomposable, and inaccessible ordinals, $cf(\alpha)$ (but not with the notation employed in the literature), and such a fundamental theorem as König's theorem (with no mention of König).

A student interested in learning set theory is likely to get more insight and inspiration from the classical texts on the subject; a working set-theorist will find H. Bachmann's *Transfinite Zahlen* more comprehensive, systematic, and a guide to the literature.

F. BAGEMIHL