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FUNCTIONS WHOSE PARTIAL DERIVATIVES ARE MEASURES

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Let x denote a generic point of euclidean N-space $R^N (N \geqq 2)$. We consider the space $\mathfrak{F}$ of all summable functions $f(x)$ such that the gradient grad f (in the distribution theory sense) is a totally finite measure. $I(f)$ denotes the total variation of the vector measure grad f. In case grad f is a function F we have

$$I(f) = \int_{R^N} | F(x) | \, dx.$$

We write H_k for Hausdorff k-measure; and fr E for the frontier of a set E. Fr E is *rectifiable* if it is the Lipschitzian image of a compact subset of R^{N-1}.

One ought to be able to determine the primitive f with greater precision than grad f, at least in certain cases. Our main result is that indeed f can be determined up to H_{N-1}-measure 0 in two (quite opposed) cases: (1) grad f is a function; (2) the range of f is a discrete set, which we may take to be the integers. More precisely, let $\mathfrak{F}_1$, $\mathfrak{F}_2$ be the sets of those $f \in \mathfrak{F}$ satisfying (1) and (2) respectively. Let $\mathfrak{F}_{01}$ be the set of all Lipschitzian functions f with compact support. Let $\mathfrak{F}_{02}$ be the set of all functions f with the following property: there exist a closed oriented $(N-1)$-polyhedron A and a Lipschitzian mapping $g(w)$ from A into R^N such that, for every $x \notin g(A)$, $f(x)$ is the degree of the mapping g at x, and $f(x) = 0$ for $x \in g(A)$. Write $J(w)$ for the Jacobian vector of $g(w)$, wherever it exists. Let Q denote the set of points $x \in g(A)$ at which there is a nonunique tangent; more precisely, we say that $x \in Q$ if there exist $w, w' \in A$ such that: (1) g is totally differentiable at w and w'; (2) $g(w)=g(w')=x$; (3) $J(w)\neq 0$, $J(w')\neq 0$; and (4) $J(w)$ and $J(w')$ do not point in the same direction.

DEFINITION. A function $f\in\mathfrak{F}_i$ is *precise* if there is a sequence $f_n\in\mathfrak{F}_{0i}$ I-convergent to f such that $\lim_n f_n(x)=f(x)$ pointwise except in H_{N-1}-measure 0.

THEOREM 1. *For $i=1$ or 2 every function $f\in\mathfrak{F}_i$ is H_N-almost everywhere equal to a precise function f'. For $i=1$ f' is uniquely determined up to H_{N-1}-measure 0. For $i=2$ f' is unique up to H_{N-1}-measure 0 if we impose the additional restriction that f_n is obtained from a mapping g_n as above for which* $\lim_n H_{N-1}(Q_n)=0$.

The idea of precise function is closely related to Aronszajn's notion of perfect functional completion. In fact:

THEOREM 2. *The class of exceptional sets for the perfect functional I-completion* [1] *of the space $\mathfrak{F}_{01}$ is the class of all H_{N-1}-null sets in R^N.*

Fuglede [6] recently treated the analogous situation when $\operatorname{grad} f\in L^p$, $p>1$. The exceptional sets turn out to be those sets E on which the Riesz potential of appropriate order of some non-negative function in L^p can be $+\infty$. Every set of Hausdorff dimension $<N-p$, and none of Hausdorff dimension $>N-p$, is exceptional. For $p=2$, considered previously by Deny and Lions [3], and Aronszajn and Smith [1], the exceptional sets are those of classical outer capacity 0 of order 2.

A set E has *finite perimeter* if its characteristic function belongs to $\mathfrak{F}_2$ (see De Giorgi [2]; in [5] I called E *Caccioppoli set*).

THEOREM 3. *Let E have finite perimeter. Then there is a sequence of open sets E_n and a set E' coincident with E except in a H_N-null set such that*: (1) fr E_n *is rectifiable for every n; and* (2) *the characteristic function of E_n converges to the characteristic function of E' in the I-norm and also pointwise except in H_{N-1}-measure* 0. *E' is uniquely determined up to H_{N-1}-measure* 0 *if we require in addition that*

$$\lim_n H_{N-1}[x\in \operatorname{fr} E_n \mid E_n \textit{ does not have an exterior normal}^1 \textit{ at } x] = 0.$$

Let E be any bounded set in R^N. Put

$$\delta(E) = \inf_f I(f),\, f\in\mathfrak{F}_{01},\, f(x)\geqq 1 \text{ for } x\in E.$$

For any set E, put

[1] In Federer's sense.

$$c(E) = \inf_{\{E_k\}} \sum_{k=1}^{\infty} \delta(E_k),\ E_k \text{ bounded},\ \bigcup E_k \supset E.$$

If we replace H_{N-1} by c, then Theorem 2 and the case $i=1$ of Theorem 1 follow easily from [1]. We need to show that

$$c(E) = 0 \text{ if and only if } H_{N-1}(E) = 0.$$

"If" is easy. To prove "only if" we first show that $\delta(E)=\delta_1(E)$, where

$$\delta_1(E) = \inf H_{N-1}(\text{fr } \pi),\ \pi \supset B,\ \pi \text{ polyhedron}.$$

Then we apply a boxing inequality recently proved by W. Gustin, which states that any polyhedron π can be covered by a finite number of cubes C_j such that

$$\sum_j H_{N-1}(\text{fr } C_j) \leqq KH_{N-1}(\text{fr } \pi)$$

where K is a constant depending only on the dimension N.

The case $i=2$ and Theorem 3 require in addition results of De Giorgi and Federer, and especially an approximation theorem for closed generalized hypersurfaces a special case of which appears in [4, p. 331].

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