ON ISOMORPHISMS OF GROUP ALGEBRAS

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With every locally compact topological group G there is associated its group algebra $L(G)$, the space of all complex Haar-integrable functions on G with convolution as multiplication. Considerable work has been done toward discovering the extent to which the algebraic structure of $L(G)$ determines G (see [**1**; **2**; **5**]), but some very specific questions have been left unanswered. For instance: Is the group algebra of the circle isomorphic to that of the torus? The theorem announced here stems from this question.

THEOREM. *The group algebra of a locally compact topological group T is isomorphic to that of the circle group C if and only if T is a direct sum $C+F$, where F is a finite abelian group.*

The proof leans heavily on that of Theorem 1 of [**4**]. In the outline below we will mainly be concerned with pointing out the changes in [**4**] which are needed to yield the stated result.

If $L(T)$ and $L(C)$ are isomorphic, then T is abelian, and the dual group Γ of T is homeomorphic to J, the group of all integers (the dual group of C) [**2**, p. 478]. Thus Γ is discrete and countable, and T is a compact abelian group with countable base.

Abelian groups will be written additively; for $x\in T$ and $\phi\in\Gamma$ the symbol (x, ϕ) will stand for the value of the character ϕ at the point x; the Haar measure on T will be denoted by m.

LEMMA 1. *Corresponding to every $E\subset T$ with $m(E)>0$, there is only a finite set of characters ϕ such that, for all $x\in E$,*

$$|1-(x,\phi)| < 1. \tag{1}$$

Note that (1) holds if and only if the real part of (x, ϕ) exceeds $1/2$. If f is the characteristic function of E and if ϕ satisfies (1), then $|\int_T(x, \phi)f(x)dx| > m(E)/2$, and the lemma follows from the Bessel inequality.

LEMMA 2. *Every infinite subset A of Γ contains an infinite subset B, such that for some $x\in T$ the inequality*

$$|1-(x,\phi)| \geqq 1 \tag{2}$$

holds for every $\phi\in B$.

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This is proved by repeated application of Lemma 1.

If now ψ is an isomorphism of $L(T)$ onto $L(C)$, ψ can be extended to an isomorphism of the measure algebras $M(T)$ and $M(C)$, and [2, p. 479] there is a one-to-one mapping α of J onto Γ such that the Fourier-Stieltjes coefficients of $\psi(\mu)$ are

$$c_n(\psi(\mu)) = \int_T (-x, \alpha(n))d\mu(x) \qquad (n \in J, \mu \in M(T)). \tag{3}$$

For $x \in T$, let e_x be the measure of mass 1 which is concentrated at x, and put $\mu_x = \psi(e_x)$. Then $c_n(\mu_x) = (-x, \alpha(n))$, and

$$\mu_x * \mu_y = \mu_{x+y} \qquad (x, y \in T). \tag{4}$$

The mapping $x \to \mu_x$ is thus an isomorphism of T into $M(C)$.

The discrete parts λ_x of μ_x also satisfy (4), and there is a mapping β of J into Γ such that

$$c_n(\lambda_x) = (-x, \beta(n)) \qquad (n \in J, x \in T); \tag{5}$$

the lemma used in Step 5 of [4] must here be applied to $C \times T$ in place of $C \times C$. Since λ_x is discrete, $c_n(\lambda_x)$ is an almost periodic function on J, for each $x \in T$. Arguing as in Step 6 of [4], we find that there is a positive integer k and a set $E \subset T$ with $m(E) > 0$, such that

$$|1 - (x, b(n))| < 1 \qquad (n \in J, x \in E), \tag{6}$$

where $b(n) = \beta(n+k) - \beta(n)$. By Lemma 1, the sequence $\{b(n)\}$ has only a finite set of values, so that the almost periodicity of $\{(x, b(n))\}$ implies that $\{(x, b(n))\}$ is actually periodic, for every $x \in T$. A compactness argument now shows that $\{b(n)\}$ is itself periodic, with period p, say. If $q = kp$, it follows that

$$\beta(n + q) + \beta(n - q) = 2\beta(n) \qquad (n \in J). \tag{7}$$

Next we put $\tau_x = (\lambda_x - \mu_x) * \lambda_{-x}$, so that

$$c_n(\tau_x) = 1 - (x, \gamma(n)) \qquad (n \in J, x \in T), \tag{8}$$

where $\gamma(n) = \beta(n) - \alpha(n)$. Since the measures τ_x are continuous,

$$\lim_{N \to \infty} \frac{1}{2N} \sum_{-N}^{N} c_n(\tau_x) = 0 \qquad (x \in T). \tag{9}$$

These averages are uniformly bounded on T, so that (9) may be integrated; combined with (8), this implies that $\gamma(n) = 0$ except possibly on a set $S \subset J$ of density 0.

Thus if S is infinite, S contains an infinite set $\{n_k\}$ such that none of the integers $n_k+1, n_k+2, \cdots, n_k+k$ belong to S, and by Lemma 2 there is an $x \in T$ and a subsequence of $\{n_k\}$, again denoted by $\{n_k\}$, such that $|c_{n_k}(\tau_x)| \geqq 1$. A subsequence of the measures

$$d\sigma_k(\theta) = e^{-in_k\theta} d\tau_x(\theta) \tag{10}$$

then converges weakly to a singular measure σ [**3**, p. 236] with $|c_0(\sigma)| \geqq 1$ but $c_n(\sigma)=0$ for all $n>0$. This is impossible, so that S is finite.

It follows that $\alpha=\pi\beta$, where β satisfies (7) and maps J onto Γ, and π is a permutation of Γ which moves only a finite number of terms; β maps each residue class mod q onto an arithmetic progression in Γ; hence Γ is finitely generated and is therefore a direct sum of a finite set of cyclic groups; since Γ is the union of a finite set of arithmetic progressions, only one of the direct summands can be infinite, so that Γ is a direct sum of J and a finite abelian group F.

This proves one half of the theorem. The converse may be proved by defining

$$\alpha(nq + k) = (n, f_k) \qquad (n \in J,\ 1 \leqq k \leqq q), \tag{11}$$

where $f_1, \cdots, f_q$ are the elements of F; it is easily verified that this induces, via (3), an isomorphism of $L(T)$ onto $L(C)$. In fact, every α of the above form $\alpha=\pi\beta$ has this property, as can be seen by an argument analogous to that on p. 50 of [**4**].

References

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