RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

ON 2-SPHERES IN 3-MANIFOLDS

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1. Let M be a connected, orientable, triangulated 3-manifold and let $\Lambda \subset \pi_2(M)$ be a sub-group which is invariant under the operators in $\pi_1(M)$. By a $\pi_1(M)$-class in $\pi_2(M)$ we mean the set of elements $\pm \xi a$, for some $a \in \pi_2(M)$ and every $\xi \in \pi_1(M)$. The invariance under $\pi_1(M)$ means that, if $a \in \Lambda$, then the entire $\pi_1(M)$-class $\{\pm \xi a\}$ is contained in Λ. A $\pi_1(M)$-class is represented by a map $S^2 \to M$, without reference to base-points or orientation. We shall describe such a map, or singular sphere, as *essential* mod Λ if, and only if, the corresponding $\pi_1(M)$-class is not contained in Λ. The terms polyhedral, piecewise linear etc., when applied to M, will refer to the piecewise affine structure which M derives from the given triangulation K. Thus a polyhedron in M is the carrier of a subcomplex, L, of a (rectilinear) subdivision of K. The polyhedron is compact if, and only if, L is a finite complex.

The main purpose of this note is to show how the proof of the (qualified) sphere theorem, due to C. D. Papakyriakopoulos [5], can be modified so as to yield a proof of:

THEOREM (1.1). *If* $\Lambda \neq \pi_2(M)$, *then* M *contains a non singular polyhedral 2-sphere which is essential* mod Λ.

On taking $\Lambda = 0$ in (1.1) we obtain the sphere theorem in full generality.

By attaching 3-cells to M we can imbed it in a space X such that Λ is the kernel of the injection $\pi_2(M) \to \pi_2(X)$. Hence it follows that (1.1) is equivalent to:

THEOREM (1.2). *If* $M \subset X$, *where* X *is a topological space, and if there is a map* $S^2 \to M$, *which is essential in* X, *then* M *contains a nonsingular, polyhedral 2-sphere which is essential in* X.

In particular if X is any orientable 3-manifold, which need not be paracompact, and if f: $S^2 \to X$ is an essential map, then every neighbourhood in X of fS^2 contains a nonsingular 2-sphere, which is essential in X and polyhedral in some paracompact, and hence triangulable [3], neighbourhood of fS^2.

If M is compact and unbounded, then $\pi_2(M)$ is a free Abelian group whose rank is 0, 1 or ∞ according as $\pi_1(M)$ has less than 2, 2 or ∞ ends [7]. Hence it follows that $\pi_2(M)\neq 0$ if M is a free product with two nontrivial factors. On the other hand if M is orientable, $\pi_2(M) \neq 0$ and $\pi_1(M)$ is not cyclic, then it is easily deduced from (1.1) that M contains an essential, nonsingular (polyhedral) 2-sphere S which separates M. In this case, therefore $\pi_1(M)$ is a nontrivial free product (see a forthcoming paper by J. W. Milnor). I hope to publish, in the Colloquium Mathematicum, a proof that, if $\pi_1(M)$ is a nontrivial free product, then M contains an essential, nonsingular 2-sphere, even if M is nonorientable (cf. [1]).

The group $Z+Z_2$ is neither cyclic nor, being Abelian, a nontrivial free product. Therefore the example $\Lambda=0$, $M=S^1\times P^2$, where P^2 is a real projective plane, shows that (1.1) is false for nonorientable manifolds.

I have been helped in the preparation of this paper by many discussions with J. W. Milnor.

2. **Proof of** (1.2). Let D_0 be a canonical singular 2-sphere in Int (M), essential in X and having the smallest (t, d)-index among all such singular 2-spheres (cf. [5, no. 19]). We build a tower over D_0, using the method and notations of [5] with certain modifications. If $\pi_1(D_0)$ is finite the tower consists of $M=M_0\supset V_0\supset D_0$. If $\pi_1(D_0)$ is infinite let $\tilde{D}_0$ be a universal cover of D_0. Let $f_0\colon G\rightarrow D_0$ be as in §10 of [5], where G is a 2-sphere, and let f_0 be lifted to $\tilde{f}_0\colon G\rightarrow\tilde{D}_0$. We identify $\pi_1(D_0)$ with the group of covering transformations of $\tilde{D}_0$ in the usual way. Clearly $\tilde{f}_0G=D^*$, say, is a fundamental region in $\tilde{D}_0$ and since D^* is compact, $\tilde{D}_0$ connected and noncompact, there is a $\tau\in\pi_1(D_0)$ such that $\tau\neq 1$, $D^*\cap\tau D^*\neq\emptyset$. Let (τ) be the sub-group of $\pi_1(D_0)=\pi_1(V_0)$ generated by τ.

We now build the tower as on p. 11 of [5] except that, if $\pi_1(D_0)$ is infinite, then:

(2.1) *M_1 is a cover of V_0 associated with (τ) and M_i is a universal cover of V_{i-1} if $i>1$,*

(2.2) *the construction terminates with the first n such that $\pi_1(V_n)$ is finite.*

Thus the tower is defined for $n\geqq 0$. If $n>0$, then $\pi_1(M_1)\approx(\tau)$, $\pi_1(V_n)$ is finite and the groups $\pi_1(V_0), \cdots, \pi_1(V_{n-1})$ are all infinite. Moreover the projection $\tilde{D}_0\rightarrow M_1$ carries $D^*\cap\tau D^*$ into a nonvacuous set of double curves in D_1. Therefore we have:

(2.3) *if $n>0$, then D_1 is singular.*

Let $n \geqq 0$ and suppose that D_n is singular. Then the components of $\dot{V}_n$ are all spheres because $\pi_1(V_n)$ is finite [6, p. 223]. We refer to No. 22 in [5] with the following modifications. The homotopies in M_0, appearing on pp. 16, 17 of [5], are to be replaced by homotopies in X. Observe that P (see the bottom of p. 16 in [5]) is such that $\pi_1(P) \approx \pi_1(V_n)$, which is finite. Therefore at the top of p. 17 in [5] we have $mK \sim 0$ in P for some $m \geqq 1$ whence the integral intersection number $sc(K, D_n) = 0$ and (22.2) of [5] follows. Moreover $\tilde{P}$, the universal cover of P, has the same homotopy type as S^3 so $\pi_2(P) \approx \pi_2(\tilde{P}) = 0$. Therefore we reach the same contradiction as on p. 17 of [5] and we conclude that D_n is nonsingular.

Since D_n is nonsingular it follows from (2.3) above that $n > 1$ if $n > 0$. In this case $\pi_1(M_{n-1}) \approx (\tau)$ or 0, according as $n = 2$ or $n > 2$. In either case $\pi_1(M_{n-1})$ is Abelian.

Assume that $n > 0$. If $H_1(V_{n-1})$ were infinite, then (23.1) in [5] would follow from [5, (11.2), (12.6)]. The last paragraph in No. 23 of [5] would then lead to a contradiction. Therefore $H_1(V_{n-1})$ is finite. Hence all the components of $\dot{V}_{n-1}$ are spheres and, by (6.1) in [5], the injection $\pi_1(V_{n-1}) \to \pi_1(M_{n-1})$ is a monomorphism. Since $\pi_1(M_{n-1})$ is Abelian, so is $\pi_1(V_{n-1})$. Therefore $\pi_1(V_{n-1}) \approx H_1(V_{n-1})$ which is finite. This contradicts (2.2). Therefore $n = 0$, D_0 is nonsingular and the proof is complete.

3. **Consequences of** (1.1). Let X be a Hausdorff space and $M \subset X$ a connected 3-manifold such that $M - \dot{M}$ is an open subset of X. Let M be orientable if X is not a 3-manifold (if X is a 3-manifold M need not be orientable).

THEOREM (3.1). *The kernel of the injection,* $i_*: \pi_1(M) \to \pi_1(X)$, *contains no element of finite order* > 1.

PROOF. The group $\pi_1(M) = \pi_1(M, x_0)$, may be identified with $\text{Lim}_{\to} \{\pi_1(C, x_0)\}$ for every compact $C \subset M$ which contains x_0. Such a C has a paracompact neighbourhood in M and it follows from the triangulation theorem that it is contained in a compact (triangulable) manifold in M. Therefore we may assume that M is compact and triangulated. If M is unbounded it is open and closed in X, whence $i_*: \pi_1(M) \approx \pi_1(X) = \pi_1(X, x_0)$. So we assume that M is bounded.

Let $1 \neq \alpha \in \pi_1(M)$, $\alpha^m = 1$ for some $m > 1$. We have to prove that $i_*\alpha \neq 1$. First let M be nonorientable, X being a manifold. Let X_1 be an orientable cover of X, let $p: X_1 \to X$ be the projection and let M_1 be the component of $p^{-1}M$ which contains the base point (in $p^{-1}x_0$) for X_1. Then M_1 is orientable. If $i_*\alpha \notin p_*\pi_1(X_1)$, then $i_*\alpha \neq 1$. If $i_*\alpha \in p_*\pi_1(X_1)$, then a loop $I \to M$, representing α and $i_*\alpha$, lifts into a loop $I \to M_1$. Therefore $\alpha = p_*' \alpha_1$, where $\alpha_1 \in \pi_1(M_1)$ and $p' = p | M_1$: $M_1 \to M$. Moreover $i_* \alpha = p_* i_*' \alpha_1$, where i': $M_1 \subset X_1$. Since $\alpha \neq 1$ and p_*, p_*' are monomorphisms it follows that $\alpha_1 \neq 1$, $\alpha_1^m = 1$ and that $i_*' \alpha_1 \neq 1$ implies $i_* \alpha \neq 1$. Therefore the theorem will follow when we have proved it for an orientable M and arbitrary X.

If G is any group let $\rho(G)$ be the minimum number of generators among all presentations of G. Thus $0 \leqq \rho(G) \leqq \infty$ and $\rho(G) = 0$ if, and only if, $G = 1$. If $G = G_1 * G_2$, a free product, then $\rho(G) = \rho(G_1) + \rho(G_2)$ [2; 4]. Let $\nu(M)$ be the number of components of $\dot{M}$ and let $\lambda(M) = \rho(\pi_1(M)) + \nu(M)$. Then $\lambda(M) < \infty$ since M is compact. If $\lambda(M) = 0$ there is nothing to prove and we proceed by induction on $\lambda(M)$. It will be enough to sketch the proof because of its similarity to that of (31.2) in [5].

Let us describe the pair (X, M) as *bad* if, and only if, kernel (i_*) contains an element of finite order > 1. Assume that the theorem is false and that, among all bad pairs, (X, M) is one with the smallest $\lambda(M)$. Then M is not aspherical, by (31.1) of [5]. Hence, and since M is bounded, it follows from (25.1) of [5] that $\pi_2(M) \neq 0$. Therefore it follows from (1.1), with $\Lambda = 0$, that M contains a nonsingular, polyhedral 2-sphere S, which is essential in M. We may assume that $S \subset M - \dot{M}$ and, since $\pi_1(M)$ contains an element of finite order > 1, that S separates M. Hence, by cutting[1] through S and filling in the holes, as in [5], we construct a pair (X', M') such that $\lambda(M') < \lambda(M)$ and (X', M') is bad (notice that $\nu(M)$, unlike $n(M)$ in [5], includes the count of the 2-spheres in $\dot{M}$). This contradiction completes the proof.

COROLLARY[2] (3.2). *If X, M are as in* (3.1) *and if $\pi_1(X)$ contains no element of finite order > 1, then $\pi_1(M)$ contains no element of finite order > 1.*

Let A be a connected subset of M which is a compact ANR (for the category of separable metric spaces). Then there is an open subset $U \subset M$ of which A is a retract. Therefore the injection $\pi_1(A) \to \pi_1(U)$ is a monomorphism and from (3.1), applied to X, U, we have:

COROLLARY (3.3). *Let X, M be as in* (3.1) *and let A be a connected, compact ANR in M. Then the kernel of the injection $\pi_1(A) \to \pi_1(X)$ contains no element of finite order > 1.*

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[1] Since S is a closed subset of X, and because X is a Hausdorff space, and $M - \dot{M}$ is open in X the cutting process can be carried out in the usual way. If X were not a Hausdorff space this would not, in general, be so and (3.1) would be false.

[2] Cf. (31.2) in [5].

Let M, A be as in (3.3), let M be orientable and let $\pi_1(A)$ contain an element, α, of finite order >1. Let f: $S^1 \rightarrow A$ be a map which represents α and let g: $S^1 \rightarrow M$ be a map such that $f \simeq g$ in M. I say that

$$A \cap gS^1 \neq \phi. \tag{3.4}$$

PROOF. Assume that $A \cap gS^1 = \phi$ and let $X = M \cup e^2$, where e^2 is an open 2-cell attached to M by the map g. Since $f \simeq g$ in M it follows that $i_* \alpha = 1$, where i: $A \subset X$. This contradicts (3.3), applied to X, $M - gS^1$, A, and (3.4) is proved.

On considering a cone, X, with a real projective plane, A, as base and $M =$ "X minus vertex" we see that (3.1), $\cdots$, (3.4) are not necessarily true if M is nonorientable. But in (3.2), (3.3), as in (3.1), M may be nonorientable provided X is a 3-manifold.

Let π be any group and G a π-module. By a set of *π-generators* for G we mean a sub-set $B \subset G$ such that every element of G is of the form $\sum_{b \in B} \xi_b b$, where ξ_b is in the integral group ring of π and $\xi_b = 0$ for almost all b.

Let $M = M_1 \cup M_2$, where M_1, M_2 are connected 3-manifolds such that $M_1 \cap M_2 = \dot{M}_1 \cap \dot{M}_2 =$ a 2-sphere.

LEMMA (3.5). *Let B_λ be a set of $\pi_1(M_\lambda)$-generators for $\pi_2(M_\lambda)$ and let ι_λ: $\pi_2(M_\lambda) \rightarrow \pi_2(M)$ be the injection* ($\lambda = 1, 2$). Then *$\iota_1 B_1 \cup \iota_2 B_2$ is a set of $\pi_1(M)$-generators for $\pi_2(M)$.*

This follows from the Mayer-Vietoris theorem, applied to a universal cover of M.

Let M be a connected, compact (possibly bounded) orientable 3-manifold.

THEOREM (3.6). *$\pi_1(M)$ has a finite set of $\pi_1(M)$-generators, whose $\pi_1(M)$-classes are represented by disjoint, nonsingular, polyhedral 2-spheres.*

PROOF. The assertion is trivial if $\pi_2(M) = 0$ so we assume that $\pi_2(M) \neq 0$. Then it follows from (1.1) with $\Lambda = 0$, that M contains an essential, nonsingular, polyhedral 2-sphere S. Let $\lambda(M)$ be as in the proof of (3.1). Clearly $\pi_2(M) = 0$ if $\lambda(M) = 0$ or if $\pi_1(M) = 1$ and $\dot{M}$ consists of a single 2-manifold, necessarily a 2-sphere. Therefore, if $\lambda(M) = 1$, then M is closed and $\pi_1(M)$ is cyclic infinite. In this case the manifold obtained from M by cutting through S is 1-connected, and the assertion follows without difficulty from the Hurewicz theorem, applied to a universal cover of M.

If $\lambda(M) > 1$ we may assume that S separates M. Then $M = M_1 \cup M_2$, $M_1 \cap M_2 = S$, where M_1, M_2 are orientable 3-manifolds. Let N_i

$=M_i \cup B_i$ $(i=1, 2)$, where B_i is a 3-dimensional ball such that $S=\dot{B}_i=B_i \cap M_i$. Clearly $\lambda(N_i)<\lambda(M)$. Moreover the kernel of the injection $\pi_2(M_i) \to \pi_2(N_i)$ is generated by the $\pi_1(M_i)$-class represented by S. Also, if $b \in B_i - S$ there is a piecewise linear isotopy of $N_i - b$ into $M_i - S$. Therefore (3.6) follows from induction on $\lambda(M)$ and (3.5).

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