THEORY OF OPERATORS

PART II. OPERATOR ALGEBRAS

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In his earliest work with operators (reported on in Part I of this article), von Neumann recognized the need for a detailed study of families of operators. Many of the subtler properties of an operator are to be found only in the internal algebraic structure of the algebra of polynomials in the operator (and its closures relative to various operator topologies) or in the action of this algebra on the underlying Hilbert space. His interest in ergodic theory, group representations, and quantum mechanics contributed significantly to von Neumann's realization that a theory of operator algebras was the next important stage in the development of this area of mathematics. The dictates of the subject itself had called for this development.

In [20], von Neumann initiated the study of the so-called "rings of operators" also called "W^*-algebras" and, most recently, "von Neumann algebras" (by Dixmier [1]). The latter term seems particularly apt, and we shall refer to them in that way.

Let us set down some notation and definitions.

DEFINITION. *A family of (bounded) operators $\mathfrak{F}$ is said to be self-adjoint when A in $\mathfrak{F}$ implies A^* is in $\mathfrak{F}$, A^* the adjoint operator to A. The 'commutant', $\mathfrak{F}'$, of $\mathfrak{F}$ is the set of those operators which commute with each operator in $\mathfrak{F}$.*

We denote by "(x, y)" the inner product of the two vectors x, y in the Hilbert space $\mathcal{H}$ and by "$\|x\|$" the "length", $(x, x)^{1/2}$, or "norm" of the vector x. If A is an operator on $\mathcal{H}$, the continuity of A is equivalent to its boundedness;

$$\|A\| = \sup \{\|Ax\| : \|x\| = 1\} < \infty,$$

and $\|A\|$ is called "the bound" or "norm" of A. With $d(A,B) = \|A - B\|$, d is a metric on the bounded operators, and the topology induced is called the "uniform" (also "norm" and "bound") topology. The weak operator topology is the topology on the bounded operators with the fewest open sets for which the mappings $A \rightarrow (Ax, y)$ of bounded operators into complex numbers is continuous, for each pair of vectors x, y in $\mathcal{H}$. The strong operator topology is the topology on the bounded operators with the fewest open sets for which the mapping

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$A \rightarrow Ax$ is continuous, for each vector x in $\mathcal{H}$ (employing the metric topology on $\mathcal{H}$). When no confusion can arise, we shall use the same symbol to denote an orthogonal projection operator and its range.

DEFINITION. *A "von Neumann algebra" is a self-adjoint algebra of operators which is closed in the weak operator topology. A "factor" is a von Neumann algebra whose center consists of the scalar multiples of its unit element.*

The many different operator topologies defined are not devices for multiplying the number of theorems one can state. They each arise critically in important situations, often being tailored to implement a particular point in a proof. It was von Neumann who recognized this technique, developed it, and used it extensively (for example, cf. [**25**]).

The key result of [**20**] is:

THE DOUBLE COMMUTANT THEOREM. *The strong closure, $\mathcal{R}$, of the algebra generated by a self-adjoint family of operators, $\mathcal{F}$, is the von Neumann algebra generated by $\mathcal{F}$, contains a unit element, E, its maximal projection, and, if E is the identity operator, I, then $\mathcal{R} = (\mathcal{F}')'$.*

The proof reduces to showing that if A is in $(\mathcal{F}')'$ then A is a strong limit point of $\mathcal{R}$ (when $\mathcal{R}$ contains I), i.e., if a positive ϵ and vectors $x_1, \cdots, x_n$ are given, we must find B in $\mathcal{R}$ such that $\|(A-B)x_i\| < \epsilon$, $i = 1, \cdots, n$. The essence of von Neumann's proof is contained in the following argument. Let $[\mathcal{R}x]$ be the closed linear manifold generated by the vectors Tx, T in $\mathcal{R}$. Clearly, each operator T in $\mathcal{F}$ leaves $[\mathcal{R}x](=E')$ setwise invariant, so that $E'TE' = TE'$. Since T^* lies in $\mathcal{F}$, $E'T^*E' = T^*E'$, and, taking adjoints, $E'T = E'TE' = TE'$, so that E' lies in $\mathcal{F}'$. Thus, by assumption, A commutes with E', and A leaves $[\mathcal{R}x]$ invariant. Since $\mathcal{R}$ contains I, $[\mathcal{R}x]$ contains x and, hence, Ax; which gives the desired result for the single vector x in place of $x_1, \cdots, x_n$. To handle the case of n vectors, von Neumann imbeds $\mathcal{R}$ and A as operators on the n-fold direct sum of $\mathcal{H}$ with itself by assigning to T the operator T_0 defined by $T_0(y_1, \cdots, y_n) = (Ty_1, \cdots, Ty_n)$, notes that appropriate hypotheses are inherited, and replaces x by $(x_1, \cdots, x_n)$.

In point of fact, von Neumann incorporated this idea in a somewhat more elaborate construction which gave him, at the same time, the existence of a unit element (projection) in $\mathcal{R}$. The existence of the unit can be obtained in an independent fashion, however. Observe that the range projection of an operator A (the projection on the closure of the range of A) is the strong limit of a sequence of polynomials in A, by spectral theory, and therefore lies in $\mathcal{R}$. The union of two projections is the range projection of their sum, whence the union of a finite family of projections in $\mathcal{R}$ lies in $\mathcal{R}$. Finally, the union of an arbitrary family of projections in $\mathcal{R}$, in particular, the family of range projections of operators in $\mathcal{R}$, is the strong limit point of the unions of finite subfamilies, and so lies in $\mathcal{R}$. The union of the range projections is clearly a unit for $\mathcal{R}$ and the maximal projection of $\mathcal{R}$.

This theorem has important uses in the study of von Neumann algebras. It is the primary technique for determining when an operator, T, associated with a von Neumann algebra $\mathcal{R}$ lies in $\mathcal{R}$. One simply checks that T commutes with $\mathcal{R}'$, or just a generating subset of $\mathcal{R}'$. Spectral theory tells us that the set of projections and the set of unitary operators in a von Neumann algebra are both generating sets. An illustration of this technique arises from the canonical (polar) decomposition of a closed, densely-defined operator, T (cf. Part I of this article). If VH is this (unique) decomposition of T (so that $H=(T^*T)^{1/2}$ and V is the unique partially isometric operator with initial space the range projection of H and range the range projection of T) and T lies in $\mathcal{R}$ (so that T, V, and H are bounded) then V and H lie in $\mathcal{R}$. By the Double Commutant Theorem and our previous comments, we need show only that V and H commute with each unitary operator U' in $\mathcal{R}'$. However, since

$$T = U'^*TU' = U'^*VU'U'^*HU,$$

we see that U'^*VU' and U'^*HU' provide another canonical decomposition of T. Uniqueness of this decomposition gives the desired commutativity and the conclusion.

In [**21**], abelian von Neumann algebras on separable Hilbert spaces are dealt with. It is shown that the operators in such an algebra are (Baire) functions of one self-adjoint operator in the algebra. The proof is basically a measure-theoretic one and entails the constructions which show that, under general conditions, a separable, nonatomic measure space is isomorphic with the unit interval under Lebesgue measure. The von Neumann result leads very quickly to canonical forms for self-adjoint maximal abelian algebras on separable Hilbert space. Each such is unitarily equivalent to the algebra of multiplications by essentially bounded measurable functions on L_2 of the measure space consisting of the unit interval with Lebesgue measure and a finite or countable number of points (equal to the number of minimal projections in the algebra), x_1, $x_2, \cdots$ where x_n has measure $1/2^n$. More specifically, if $\mathcal{A}$ is the algebra, E the union of its minimal projections, and there are n such projections (where n is possibly $\aleph_0$), then $\mathcal{A}$ is the direct sum of $\mathcal{A}E$, the algebra of all diagonal matrices on the n-dimensional Hilbert space $E(\mathcal{H})$ relative to an orthonormal basis for $E(\mathcal{H})$ obtained by choosing a unit vector in the range of each minimal projection, and $\mathcal{A}(I-E)$ which is unitarily equivalent to the algebra of operators, L_f, defined on $L_2(0, 1)$ (with Lebesgue measure) by $L_f(g)=fg$, provided $I-E\neq 0$.

Recognizing the need for a further and more detailed study of the weakly-closed, self-adjoint operator algebras before much progress could be made in the rapidly developing theory of group representations of infinite groups (locally compact groups, in general, and Lie groups, in particular), von Neumann undertook this project with F. J. Murray in 1935. This research was to lead to the important papers constituting the "Rings of Operators" series [**17**; **18**; **19**; **22**]. The hope that this study might provide an adequate framework for the mathematical formalization of quantum mechanics was an added incentive. The resolution of this hope lies in the future, but a strong case can be made for the cogency of these methods in that formalization. Moreover, the study has reflected back on physics through its use in the theory of representations of groups by unitary operators on a Hilbert space.

Murray and von Neumann saw at once that the factors were the basic constituents in the theory of von Neumann algebras and proceeded directly to the analysis of this special class in the first paper [**17**] of the series. The prime example of a factor is the algebra of all bounded operators on a Hilbert space. It must be noted that all evidence from classical mathematical phenomena indicates that, algebraically speaking, these are the only factors—the factor definition applied to finite-dimensional algebras yields algebras isomorphic to $n\times n$ complex matrices. Indeed, in any case, if the factor has a minimal projection (minimal idempotent) it is algebraically isomorphic to the algebra of all bounded operators on some Hilbert space. One might suspect that it is automatically the case that minimal projections exist (as indicated by the classical situation). This turns out not to be so, but the construction of factors for which it is false is no simple matter. Less powerful mathematicians, might have added the assumption that minimal projections exist, at an early stage of the study, thereby characterizing the algebra of bounded operators and bypassing the theory which is the vital force in abstract operator theory.

Murray and von Neumann took a different path. Rather than search for projections of a particular size in the factor (viz., the minimal projections) they developed a general theory of the relative sizes of projections in a factor—the so-called "comparison theory for projections." The basic idea was to consider two projections E and F as having the same size when a partially isometric operator V in the factor $\mathcal{R}$ has F for its final space and E for initial space—algebraically, $V^*V=E$, $VV^*=F$ (this defines a partial isometry, V)—geometrically, V maps the range of E isometrically onto that of F and is 0 on $I-E$. Two such projections are said to be "equivalent" (written: $E\sim F$), and, of course, $E(=V^*V)$ and $F(=VV^*)$ must lie in $\mathcal{R}$, if they are equivalent relative to $\mathcal{R}$. Making use of equivalence with subprojections in the natural way, Murray and von Neumann introduced a partial ordering, $\prec$ and $\precsim$, on the projections, with the usual notational usage. It is trivial to show that equivalence is preserved on orthogonal sums of projections, from which a Cantor-Bernstein result is straightforward. All of this was defined, and relevant for von Neumann algebras (as distinguished from the special case of factors). The crucial fact, valid for factors alone, states that this partial ordering is total in a factor—each pair of projections is comparable relative to comparison by partial isometries in the factor. In fact, for any von Neumann algebra, the polar decomposition of an operator in it provides us with a partial isometry in the algebra (as we noted earlier) which has about the same mobility properties on vectors (in getting them from space to space) as the original operator does. Now, in a factor, a proper subspace which is not moved into contact with all other subspaces by the operators of the factor, and hence its partial isometries, provides us with a proper invariant subspace, the projection on which lies in the center of the factor. Thus, in a factor, each pair of nonzero subspaces has a pair of equivalent proper subspaces. A measure theoretic exhaustion argument completes the proof of comparability in factors. Briefly, the high noncommutativity in factors provides high mobility of subspaces and this yields comparability of projections.

Infiniteness (and finiteness) of projections is defined by equivalence with proper subprojections in the usual way; and the standard facts are readily proved—with one notable exception. The finiteness of the union of two finite projections (even in the case where they are orthogonal) required a delicate and involved proof, which Murray and von Neumann supplied with characteristic power and ingenuity. This finiteness entails an analysis of the subspaces of the union, and such subspaces may be quite unrelated to the original spaces (having intersection (0) with both, for example). Much of the comparison theory can be developed by analogy with the theory of sets and cardinals, to which it is closely related. The analogy is not perfect, however, as in the situation just described (a subset of a union of sets being the union of its intersection with each of the sets); and this imperfection generates the major difficulties of the subject.

Prolonged scrutiny of the difficult points in the theory developed by Murray and von Neumann, carried out under the greatly reassuring circumstances of the validity of the result in question being known, has led to several simplifications of proofs. The finiteness result is a case in point. In [14], Kaplansky effects the proof in the following steps. Each infinite projection E is the sum of two projections equivalent to E. (It is trivial that equivalence preserves finiteness.) The essence of the problem is the analysis of a subprojection of a sum of orthogonal projections in terms of the summands, and it is shown that, with $G \leqq E+F$, either $G \precsim E$ or $E+F-G \precsim F$. Thus, with E and F finite, if $E+F$ is infinite and G is chosen as a subprojection of $E+F$ equivalent to $E+F$ and $E+F-G$, $G \precsim E$ contradicts the finiteness of E and $E+F-G \precsim F$ that of F. It follows that $E+F$ is finite. The union of finite projections is easily reduced to the orthogonal case. The polar decomposition applied to $E(I-F)$ yields the equivalence of $(E \vee F)-F$ and $E-(E \wedge F)$ (where "$\vee$" and "$\wedge$" denote union and intersection for projections). This result applied to G and F leads to $G \precsim E$, when $G \wedge F \precsim (E+F-G) \wedge E$, and applied to $E+F-G$, E leads to $E+F-G \precsim F$, when $(E+F-G) \wedge E \precsim G \wedge F$, where $G \leqq E+F$, with E and F orthogonal.

Having established these results in comparison theory, Murray and von Neumann introduce their dimension function on projections in the factor $\mathfrak{R}$. This is a function D mapping projections into the nonnegative reals and ∞ such that $D(E)>0$, if $E \neq 0$; $D(E)=D(F)$, *iff* $E \sim F$; and $D(E+F)=D(E)+D(F)$, if $EF=0$. These properties determine D up to a positive multiple; and D has the further properties, $D(0) = 0$; $D(E) \lesseqqgtr D(F)$ if and only if $E \lesssim\!\!\!> F$; $D(\sum_\alpha E_\alpha) = \sum_\alpha D(E_\alpha)$, if $\{E_\alpha\}$ is an orthogonal family; and $D(E)+D(F) = D(E \vee F)+D(E \wedge F)$, this last follows from the equivalence of $(E \vee F)-F$ and $E-(E \wedge F)$. To define D, choose any nonzero projection E in $\mathfrak{R}$, and let $D(E)=1$. A projection, F, is "rational" if F and E are the sum of m and n projections, respectively, in $\mathfrak{R}$, each equivalent to a projection, G; and let $D(F)=m/n$. For an arbitrary projection, G in $\mathfrak{R}$, let $D(G)$ be the supremum of the dimensions of rational projections contained in G. It is not difficult to show that D, as defined, is single-valued, with the properties noted.

The essential uniqueness of D provides the algebraic invariant by which a gross separation of the class of factors into distinct algebraic types can be effected. The nature of the range of D is the vital indicator. Normalizing D so that it is 1 on I, the identity operator, when I is finite—the possibilities for the range of D, on particular factors are: $\{1, \cdots, n\}$, $n=1, \cdots, \infty$—in this case, the factor is algebraically isomorphic to all bounded operators on n-dimensional Hilbert space and is said to be "of type I_n";

$[0, 1]$, the closed interval—in this case, there are no minimal projections, and I is finite—the factor is said to be "of type II_1";

$[0, \infty]$—as above, with I infinite—the factor is said to be "of type II_∞";

$\{0, \infty\}$—each nonzero projection is infinite—the factor is said to be "of type III."

As we remarked, the existence of factors of types II and III was not at all apparent. Murray and von Neumann constructed a class of examples of factors of types II_1 and II_∞ with the aid of ergodic theory. It should be recalled that abstract measure theory and ergodic theory were in their infancy at the time of these investigations, and that matters which we handle with ease and assurance today required careful verification at that time.

Let $(S, \mathcal{S}, m)$ be a measure space, with $\mathcal{S}$ the measurable sets and m the measure; and let G be a group of measure preserving transformations of S, which acts freely and ergodically (i.e., $m(\{s: g(s) = s\}) = 0$, for each g in G different from e; and if S_0 in $\mathcal{S}$ is such that $m[g(S_0) - S_0] = 0$, for each g in G then $m(S_0)$ is 0 or 1). Each g in G induces a unitary transformation U_g of $L_2(S, m)$ defined by $(U_g(a))(s) = a(g(s))$; and each essentially bounded measurable function, b, on S induces a multiplication operator, L_b, defined by $L_b(a) = ba$ (the collection of such L_b is a maximal abelian algebra). Let $\mathcal{H}$ be the direct sum of as many copies of $L_2(S, m)$ as there are elements in G (we must deal with G countable and $L_2(S, m)$ separable if we wish to remain in the separable case). The examples consist of infinite matrices, with rows and columns indexed by G, whose entries are operators on $L_2(S, m)$—the matrix operating in the usual way on $\mathcal{H}$. Let U_g^0 be the matrix whose only nonzero entry in the h column is U_g in row gh, and let L_b^0 be the matrix with zeros off the diagonal and L_b as each diagonal entry. The von Neumann algebra, $\mathcal{R}$, generated by the U_g^0 and the L_b^0 is a factor, as can be shown by a computation employing the ergodicity and freeness of the action of G upon S. Each operator in $\mathcal{R}$ has all its diagonal entries equal to L_b, for some b depending on the operator. With E a projection, the associated b is real and non-negative almost everywhere on S, so that $\int b\, dm$ is defined (possibly ∞). The mapping from E to $\int b\, dm$ is readily seen to have the properties of a dimension function on $\mathfrak{R}$ and, so, is the dimension function (up to a positive scalar multiple). Considerations of the range of this function lead immediately to the conclusion that if S has an atom, it is totally atomic, and $\mathfrak{R}$ is of type I—finite if $m(S)$ is finite, infinite otherwise; if S has no atoms, $\mathfrak{R}$ is of type II, finite or infinite as before. This example may also be viewed as the algebra of functions from the group to $L_2(S, m)$ which have square summable norms over the group and which by convolution action on the Hilbert space of all such functions, the convolution involving the group action on S, as with U_g, yield a bounded operator.

In [22], the third paper of the series, von Neumann produces examples of factors of type III, by considering groups of transformations which preserve measurability and sets of measure 0 but admit only the trivial invariant measure. The construction is just as above with the exception that U_g is defined by $(U_g a)(s) = f_g^{1/2}(s)a(g(s))$, where f_g is the Radon-Nikodym derivative of the translated measure m_g (defined by $m_g(S_0) = m[g(S_0)]$) with respect to m. The essence of the construction lies in the fact that a finite nonzero projection in the resulting factor would yield, together with the dimension function, a nontrivial invariant measure. Concrete examples of a II_1, II_∞, and III are obtained by taking for G the group generated by rotation of the circle through an irrational angle and for S the circle under Lebesgue measure, in the first case; for G all rational translations of the real line and for S the real line under Lebesgue measure, in the second case; for G the group of transformations $x \rightarrow \alpha x + \beta$, with α and β rational, $\alpha \neq 0$, of the real line and for S the real line under Lebesgue measure, in the third case.

In the examples of factors of type II_1 just constructed the mapping $T: A \rightarrow \int b\, dm$, is defined, for all operators in $\mathfrak{R}$—not just projections—since $m(S)$ is finite and b is essentially bounded, and has the properties: $T(A) > 0$, if $A > 0$ (recall that $A \geqq 0$ means $(Ax, x) \geqq 0$, for each x); $T(AB) = T(BA)$; T is linear, $T(I) = 1$. In an arbitrary factor of type II_1, a function such as T, defined just on the self-adjoint operators, with $T(AA^*) = T(A^*A)$ replacing $T(AB) = T(BA)$ and linearity assumed for commuting self-adjoint operators alone, is unique. Indeed, the restriction of T to the projections of $\mathfrak{R}$ is the normalized dimension function, from which, linearity of T on commuting self-adjoint operators and an easily proved continuity yield $T(A) = \int \gamma dD(E_\gamma)$, where $\{E_\gamma\}$ is the spectral resolution of the self-adjoint operator A in $\mathfrak{R}$. This provides us with a definition of T in the general case and the resulting function has the restricted properties $T(AA^*) = T(A^*A)$ and linearity on commuting self-adjoint operators in $\mathcal{R}$ (as well as $T(A)>0$ if $A>0$ and $T(I)=1$). This function, in the case of finite factors of type I, is the familiar trace function (normalized to be 1 at I). Murray and von Neumann call T the trace, in [**17**], and observe, as above, that it has the properties $T(AB)=T(BA)$ and unrestricted linearity in all the examples they construct, when it is extended to all operators in $\mathcal{R}$ by means of the unique decomposition of an arbitrary operator in $\mathcal{R}$ as the sum of a self-adjoint and skew-adjoint operator in $\mathcal{R}$. The property $T(AB)=T(BA)$ hinges on the unrestricted additivity of the extended trace, and most of the second paper of the series, [**18**], is devoted to a remarkably clever and intricate proof of this fact. The great difficulty stems from the inability to relate the spectral resolution of $A+B$ to that of A and of B, when A and B are self-adjoint operators which do not commute. Although several other natural methods for introducing a trace present themselves, this proof withstood essential simplification until recently. In [**1**; **9**], the basic "approximate local additivity" argument of Murray and von Neumann is employed in such a way as to give a simplified proof. The problem is easily reduced to that of finding a linear functional, f, on $\mathcal{R}$ such that $f(I)=1$ and $f(A)\geqq 0$, when $A\geqq 0$, (f is called "a state of $\mathcal{R}$") with the property $f(AA^*)=f(A^*A)$. By symmetry, it suffices to find f such that $f(AA^*)\leqq f(A^*A)$, for each A in $\mathcal{R}$, and, by compactness of the unit sphere in the dual to $\mathcal{R}$, to find f_n such that

$$f_n(AA^*) \leqq [(n+1)/n]f_n(A^*A),$$

for each positive integer n. If this can be accomplished for each A in $E\mathcal{R}E$, where E is a projection in $\mathcal{R}$ of dimension $1/m$, then, decomposing I as a sum of m projections equivalent to E and transforming the restriction of f_n to $E\mathcal{R}E$ by means of the resulting partial isometries, we obtain m functionals whose sum has the desired property on $\mathcal{R}$. Writing g in place of f_n and $\mathcal{R}$ in place of $E\mathcal{R}E$, it suffices to find g so that

$$D(F) \leqq g(F) \leqq (n+1)D(F)/n,$$

for each projection F, since then,

$$g(F) \leqq (n+1)D(G)/n \leqq (n+1)g(G)/n,$$

where G is equivalent to F, and the spectral theorem allows us to conclude this second inequality with a pair of positive self-adjoint operators in $\mathcal{R}$ which are unitarily equivalent via a unitary operator in $\mathcal{R}$ replacing F and G. In particular, the polar decomposition of A shows that AA^* and A^*A is such a pair, for A in the finite factor $\mathcal{R}$.

Multiplying f_n by a positive scalar does not change its properties, so that we may drop the restriction on f_n and g that they be 1 at I. To find g satisfying the first inequality, Murray and von Neumann take the state ω_x of $\mathcal{R}$ defined as $\omega_x(A) = (Ax, x)$, for x a fixed vector of length 1, and prove by a clever but short exhaustion argument that a nonzero projection such as E can be found, with g taken as a positive scalar multiple of ω_x. States of $\mathcal{R}$ such as ω_x are called "vector states" of $\mathcal{R}$. They are special cases of the "normal" (or "completely additive") states of $\mathcal{R}$—those states ω such that $\omega(\sum_\alpha E_\alpha) = \sum_\alpha \omega(E_\alpha)$, for each orthogonal family $\{E_\alpha\}$ of projections in $\mathcal{R}$. The trace, T, is a normal state, and Murray and von Neumann show that, in the presence of a separating vector for $\mathcal{R}$ (i.e., a vector, y, such that A in $\mathcal{R}$ is 0 if $Ay=0$), T is a vector state. We know, more generally today, that, in the presence of a separating vector, each normal state of a von Neumann algebra is a vector state [**2**; **3**; **7**]. In any event, a normal state is a convex linear sum (possibly infinite) of vector states, and if the algebra is a finite factor, each normal state is a finite convex sum.

The essence of the argument which establishes these results concerning normal states of von Neumann algebras is the following. If a vector state majorizes another state it too is a vector state—an easy consequence of the representation of bounded, conjugate-bilinear functionals on a Hilbert space as $x, y \rightarrow (Ax, y)$. Using the vector state ω_x and the given normal state as in the approximate local additivity argument of Murray and von Neumann together with some refined polarization techniques completes the proof.

In [**18**; **19**], Murray and von Neumann devote considerable space to the reduction of the spatial or unitary classification problem to the algebraic problem. Specifically, in [**18**], they show that each isomorphism between finite factors can be implemented by a unitary transformation of the underlying Hilbert spaces provided that they have equal coupling constants. In more detail, if $\mathcal{R}$ is our von Neumann algebra with commutant $\mathcal{R}'$ and x is a vector in $\mathcal{H}$, the underlying Hilbert space, then $[\mathcal{R}x]$, being invariant under the self-adjoint family $\mathcal{R}$, commutes with $\mathcal{R}$ (i.e., its orthogonal projection does) and, so, lies in $\mathcal{R}'$. Similarly, $[\mathcal{R}'x]$ lies in $\mathcal{R}$. If $\mathcal{R}$ is a factor, we can form $D([\mathcal{R}'x])$ and $D'([\mathcal{R}x])$, where D and D' are the normalized dimension functions on $\mathcal{R}$ and $\mathcal{R}'$, respectively. The ratio,

$$D([\mathcal{R}'x])/D'([\mathcal{R}x]),$$

is independent of the nonzero vector x chosen and is finite in case $\mathcal{R}$ and $\mathcal{R}'$ are finite—it is called the coupling constant. Roughly speaking, the coupling constant compares the relative sizes of a factor and its commutant. The unitary implementation theorem above says that two isomorphic finite factors are unitarily equivalent if and only if they bear the same relative size to their commutants. In [**19**], Murray and von Neumann carry these results over to a factor $\mathcal{R}$ in all cases except $\mathcal{R}$ of type III and the case of $\mathcal{R}$ of type II_∞, $\mathcal{R}'$ of type II_1. For factors of type III, Griffin [**8**] noted that, in the separable case, every isomorphism between factors of type III can be implemented by a unitary transformation—making use of results of Dye [**3**]. The type II_∞, II_1 case is settled by the result [**10**] that such an isomorphism can be implemented by a unitary transformation if and only if it carries a maximal cyclic projection (a maximal projection of the form $[\mathcal{R}'(x)]$) in one factor onto another such projection. (There exist isomorphisms which do not do this.) These later results make use of the new techniques of normal states, and, indeed, using these techniques, the whole spatial analysis of factors by Murray and von Neumann has been simplified and developed to handle the general von Neumann algebra [**2**; **4**; **7**, **8**; **11**; **13**; **27**].

The basic fact used by Murray and von Neumann to prove the existence of the coupling constant, [**17**, Lemma 9.3.3], applies to the general von Neumann algebra (as does their proof) and is a crucial step in the spatial analysis of these algebras. It states that for nonzero vectors x and y, $[\mathcal{R}x] \underset{>}{\overset{<}{\sim}} [\mathcal{R}y]$ is equivalent to $[\mathcal{R}'x] \underset{>}{\overset{<}{\sim}} [\mathcal{R}'y]$ (the relations $\sim$, $<$, $>$ understood relative to the appropriate von Neumann algebra, of course). It is easily seen that a subprojection of a cyclic projection is itself cyclic—from which, it follows quickly that the proof of the stated result reduces to proving the case: $[\mathcal{R}x] \sim [\mathcal{R}y]$ implies $[\mathcal{R}'x] \sim [\mathcal{R}'y]$. Suppose, for the moment, that we have the result. The mapping $D([\mathcal{R}'x]) \to D'([\mathcal{R}x])$ is single-valued, then, for if $D([\mathcal{R}'x]) = D([\mathcal{R}'y])$, we have $[\mathcal{R}'x] \sim [\mathcal{R}'y]$, so that $[\mathcal{R}x] \sim [\mathcal{R}y]$; and $D'([\mathcal{R}x]) = D'([\mathcal{R}y])$. In the same way the mapping is monotone increasing, and without difficulty, it can be shown to be additive, when all terms are defined. The existence of the coupling constant now follows.

The proof which Murray and von Neumann supply for Lemma 9.3.3 of [**17**] makes use of an auxiliary result [**17**, Lemma 9.2.1], which describes each vector in $[\mathcal{R}x]$ as the result of applying a closed densely-defined (unbounded) operator, T, which commutes with each operator in $\mathcal{R}'$ (we say T is "affiliated with $\mathcal{R}$") to x and then an operator in $\mathcal{R}$ to Tx. This result requires a construction of

Friedrichs [5] and, as might be expected, makes use of the theory of unbounded operators which von Neumann helped to found (cf. Part I). It is of considerable interest in itself but may be avoided and the proof of Lemma 9.3.3 simplified by making use of the technique of normal states. In fact, defining "the carrier of the normal state ω of $\mathcal{R}$" to be the complement of the union of all projections (the maximal projection) in $\mathcal{R}$ annihilated by ω, we prove at once that if the carrier of ω is contained in that of the vector state ω_x (which is easily seen to be $[\mathcal{R}'x]$) then ω is a vector state. An easy argument shows that if ω has the same carrier as ω_z, and $[\mathcal{R}z]=\mathcal{H}$, then $\omega=\omega_x$, with $[\mathcal{R}x]=\mathcal{H}$. (This need not be the case for each x such that $\omega=\omega_x$, a careful choice must be made.) One then extends this result slightly to the case where ω and ω_z are assumed only to have the same carrier and concludes the existence of x such that $\omega=\omega_x$ with $[\mathcal{R}x]=[\mathcal{R}z]$. For Lemma 9.3.3, assume that $[\mathcal{R}'x]\sim[\mathcal{R}'y]$ and that V is a partial isometry in $\mathcal{R}$ effecting the equivalence. Then

$$[\mathcal{R}'Vx] = V[\mathcal{R}'x] = [\mathcal{R}'y],$$

and

$$[\mathcal{R}Vx] \subseteq [\mathcal{R}x] = [\mathcal{R}V^*Vx] \subseteq [\mathcal{R}Vx],$$

i.e., $[\mathcal{R}Vx]=[\mathcal{R}x]$. Replacing x by Vx, we may assume that $[\mathcal{R}'x]=[\mathcal{R}'y]$. In this case, by the extended carrier result just noted, $\omega_y=\omega_z$, for some z such that $[\mathcal{R}z]=[\mathcal{R}x]$. The mapping, $Az\to Ay$, A in $\mathcal{R}$, is isometric, since $\omega_y=\omega_z$, and extends to an isometric mapping of $[\mathcal{R}z](=[\mathcal{R}x])$ onto $[\mathcal{R}y]$. This mapping, extended to $\mathcal{H}$ by defining it as 0 on the complement of $[\mathcal{R}z]$, is a partially isometric operator V' in $\mathcal{R}'$, so that $[\mathcal{R}x]\sim[\mathcal{R}y]$ in $\mathcal{R}'$.

The result which provides the fundamental building block for the spatial analysis of von Neumann algebras states that an isomorphism between von Neumann algebras $\mathcal{R}_1$ and $\mathcal{R}_2$ for which there exist vectors x and y, respectively, such that $[\mathcal{R}_1x]=\mathcal{H}_1=[\mathcal{R}_1'x]$ and $[\mathcal{R}_2y]=\mathcal{H}_2=[\mathcal{R}_2'y]$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the underlying Hilbert spaces for $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively, can be implemented by a unitary transformation of $\mathcal{H}_1$ onto $\mathcal{H}_2$. A vector such as x is said to be "a generating vector for $\mathcal{R}_1$." It is trivial to show that a generating vector for $\mathcal{R}$ is a separating vector for $\mathcal{R}'$ and conversely, so that x is both generating and separating for $\mathcal{R}_1$ and $\mathcal{R}_1'$. If γ is the (adjoint-preserving) isomorphism, define the state ω of $\mathcal{R}_1$ by: $\omega(A)=(\gamma(A)y, y)$, and note that ω is a normal state with carrier I, since γ is an isomorphism and y is separating for $\mathcal{R}_2$. From our previous remarks, there is a vector z in $\mathcal{H}_1$ such that $\omega=\omega_z$, $[\mathcal{R}_1z]=\mathcal{H}_1$ (and $[\mathcal{R}_1'z]=\mathcal{H}_1$, since ω has carrier I). The transformation, U_0, of the dense subset, $\mathcal{R}_1z$, of $\mathcal{H}_1$ onto the dense subset, $\mathcal{R}_2y$, of $\mathcal{H}_2$ defined by: $U_0Az=\gamma(A)y$, is single-valued (since z is separating), linear, and isometric, since

$$||Az||^2 = (A^*Az, z) = \omega(A^*A) = (\gamma(A^*A)y, y) = ||\gamma(A)y||^2 = ||U_0Az||^2.$$

Thus U_0 has a unique isometric (unitary) extension, U, mapping $\mathcal{H}_1$ onto $\mathcal{H}_2$. Now,

$$UAU^{-1}[\gamma(B)y] = UAU^{-1}(UBz) = U(ABz) = \gamma(AB)y = \gamma(A)[\gamma(B)y],$$

whence the two bounded operators, UAU^{-1} and $\gamma(A)$, agree on the dense subset, $\mathcal{R}_2y$ of $\mathcal{H}_2$; so that U implements γ.

In [**18**], Murray and von Neumann establish, as their basic tool in the spatial analysis of factors, the unitary implementation result above for factors of type II_1 with coupling constant 1. This is precisely the case of a separating and generating vector for the factor (and its commutant). In this case, the separating and generating vector may be chosen as a trace vector, i.e., a vector x such that ω_x is the trace on the factor (and its commutant). If such vectors are chosen for x and y, in the proof above, no adjustment of ω need be made, for $\omega=\omega_x$, since γ preserves trace (from the characterizing properties of the trace). From this observation, Murray and von Neumann were able to establish their unitary implementation result and go on to the spatial analysis of factors without the benefit of the normal state results. They were limited, however, to situations which could be related to a trace, i.e., to factors not of type III. It is a curious circumstance that their approach to the spatial analysis of factors pointed the way to the spatial analysis of general von Neumann algebras and, at the same time, diverted attention from the fundamental concepts by its reliance on the trace.

It is appropriate to note, at this point, the result which relates factors of type II_∞ to those of type II_1. This relation is studied in detail in [**19**]. If $\mathcal{R}$ is a factor of type II_∞ the identity projection, I, is infinite and the sum of a mutually-orthogonal, infinite family of equivalent finite projections, $\{E_\alpha\}$, in $\mathcal{R}$. With E_0 in $\{E_\alpha\}$, let $E_{0\alpha}$ be a partial isometry in $\mathcal{R}$ with initial space E_α and final space E_0; and let $E_{\alpha\beta}$ be $E_{0\alpha}^*E_{0\beta}$, a partial isometry in $\mathcal{R}$ with initial space E_β and final space E_α. (The set $\{E_{\alpha\beta}\}$ forms a family of matrix units for $\mathcal{R}$.) The commutant of $\{E_{\alpha\beta}\}$ in $\mathcal{R}$, $\{E_{\alpha\beta}\}'\wedge\mathcal{R}$, is a factor, $\mathcal{R}_0$, of type II_1, easily seen to be isomorphic to $E_\alpha\mathcal{R}E_\alpha$, for each α; and, without difficulty, it can be shown that $\mathcal{R}$ is unitarily equivalent to (hence isomorphic with) the algebra of $\aleph\times\aleph$ matrices over $\mathcal{R}_0$ acting in the standard manner on the direct sum of $E_0(\mathcal{H})$ with itself $\aleph$ times, which give bounded operators by such action, where $\aleph$ is the cardinality of $\{E_\alpha\}$. Another way of stating this result is to say that each factor of type II_∞ is the Kronecker product of one of type II_1 with one of type I_∞.

We have remarked that the existence of factors of types other than I is a phenomenon not to be expected from classical evidence. The further separation of the class of factors into the algebraic types I_n, II_1, II_∞, and III by means of a natural algebraic invariant, and the proof of the existence of factors of each type, raises the crucial question of whether or not this completes the algebraic classification —are all factors of a given type isomorphic (separable case—this is true for factors of type I). This question occupied much of the attention of Murray and von Neumann during the years of their research on operator algebras. They succeeded in answering this question in [19]—an answer which gave clear evidence of the complexity of the area they had penetrated. In [19], Murray and von Neumann produced two nonisomorphic factors of type II_1. This is done with the aid of a new class of examples of factors of type II_1 (different from those obtained from the ergodic theory construction of [17], we have described). We emphasize that the construction is different though it may well be the case that factors arising from these different constructions are isomorphic. It is not known, to this day, whether or not all factors of type II_1 arise from a given one of the constructions (or from both).

This class of examples is obtained as the group algebras of certain (discrete) infinite groups. Let G be a countably-infinite group with unit element, e, and let $\mathcal{H}$ be the Hilbert space of square-summable functions on G with the usual inner product (l_2). For f and g in $\mathcal{H}$, define:

$$(f * g)(a) = \sum_{b \in G} f(ab^{-1})g(b) = \sum_{c \in G} f(c)g(c^{-1}a)$$

(the last equality obtained by taking $b = c^{-1}a$ and noting that $c \rightarrow c^{-1}a$ is a 1-1 mapping of G onto G). With f and g in l_2, the sums, in question, converge absolutely, by Schwarz's inequality. Let L_f be the transformation defined on $\mathcal{H}$ by: $L_f(g) = f * g$; similarly, let R_f be defined by: $R_f(g) = g * f$. With a in G, let a' be the function which is 1 at a and 0 elsewhere. It is trivial to verify that $L_{a'}$ and $R_{a'}$ are unitary operators on $\mathcal{H}$. The easily proved result that, for T a bounded operator on $\mathcal{H}$ and f in $\mathcal{H}$, if $(Tb', a') = (f * b', a')$, for each a and b in G, then $L_f = T$, eliminates convergence difficulties and leads quickly to the following facts:

(1) With L_f and L_g bounded operators; $L_f + L_g = L_{f+g}$, $L_f L_g = L_{f\,g}$,

$\alpha L_f = L_{\alpha f}$, $L_f^* = L_{f^*}$ where $f^*(a) = \bar{f}(a^{-1})$, $L_{e'} = I$, if $L_f = L_g$ then $f = g$.

(2) The sets $\mathcal{L}_G$ and $\mathcal{R}_G$ of all L_f and R_f, respectively, f in $\mathcal{H}$, which are bounded operators on $\mathcal{H}$ are self-adjoint algebras.

(3) $\mathcal{L}_G' = \mathcal{R}_G$ and $\mathcal{R}_G' = \mathcal{L}_G$, so that $\mathcal{L}_G$ and $\mathcal{R}_G$ are von Neumann algebras, and are generated by $\{L_{a'}\}$, $\{R_{a'}\}$, respectively.

The von Neumann algebra $\mathcal{L}_G$ is a factor if and only if each conjugate class of G other than $\{e\}$ is infinite. Indeed, if C is a finite conjugate class in G and f is the function which is 1 on C and 0 elsewhere on G then $L_f L_{a'} = L_{a'} L_f$, so that L_f lies in the center of $\mathcal{L}_G$, and $\mathcal{L}_G$ is a factor if and only if $L_f = \alpha I$, i.e., $C = \{e\}$. If each conjugate class other than $\{e\}$ is infinite and L_f lies in the center of $\mathcal{L}_G$, then $L_{a'^{-1}} L_f L_{a'} = L_f$, from which $f(aca^{-1}) = f(c)$, for each a and c in G. Thus, with G_c the conjugate class of c,

$$\sum_{a \in G_c} |f(c)|^2 = \infty |f(c)|^2 \leqq \sum_{b \in G} |f(b)|^2 < \infty,$$

when $c \neq e$, so that $f(c) = 0$, for such c; and $L_f = \alpha I$, i.e., $\mathcal{L}_G$ is a factor.

Defining $T(L_f)$ to be $f(e)$ for L_f in $\mathcal{L}_G$, we deduce at once that T is the normalized trace on $\mathcal{L}_G$, which is, accordingly, a finite factor. Not being finite-dimensional, $\mathcal{L}_G$ is of type II_1. Specific examples of groups giving rise to factors of type II_1 are obtained by considering G_p, the group of permutations of the integers each leaving fixed all but a finite number of integers; and G_2 the free group on two generators. The factors arising from these two groups are, in fact, ones which Murray and von Neumann show to be nonisomorphic. This is accomplished by the following argument. If $\mathcal{R}$ is a factor of type II_1 and T its normalized trace; $A, B \rightarrow T(B^*A)$ defines an inner product on $\mathcal{R}$ relative to which $\mathcal{R}$ becomes a pre-(incomplete) Hilbert space. We denote the norm of A in this inner product, $T(A^*A)^{1/2}$, by $[[A]]$, and call the topology induced by the metric $A, B \rightarrow [[A-B]]$ "the metric topology on $\mathcal{R}$." (Relative to convergence in the metric topology, it is easy to show that, for each L_f of the class of examples just described, $L_f = \sum_{a \in G} f(a) L_{a'}$, independently of the order of summation over G.) Murray and von Neumann now define a weak commutativity property for factors in terms of this metric as follows. The factor $\mathcal{R}$ is said to have property Γ if, for each finite set of operators $\{A_1, \cdots, A_n\}$ in $\mathcal{R}$ and positive ϵ, it is possible to find a unitary operator U in $\mathcal{R}$ such that

$$[[UA_k - A_k U]] < \epsilon; \qquad k = 1, \cdots, n,$$

and $T(U) = 0$ (without this condition, one could always choose I for U). If the group G gives rise to a factor $\mathcal{L}_G$ and has the additional property that, for any finite subset of G, some group element distinct from e can be found which commutes with each of them, then $\mathfrak{L}_G$ has property Γ. This is the case for G_p. Indeed, each A_k can be approximated metrically to within $\epsilon/2$ by a finite linear combination of $L_{a'}$'s, and if c, different from e, commutes with each a' (for all k; a finite subset of G) then $T(L_{c'})=0$; and $L_{c'}$ is a unitary operator satisfying the inequality of property Γ.

That $\mathfrak{L}_{G_2}$ does not have property Γ is a somewhat more difficult matter—though, the freeness of G_2 might lead one to suspect this. Suppose, in fact, that G is a factor group containing a subset S and elements a_1, a_2, a_3 such that $S \vee a_1Sa_1^{-1} = G - \{e\}$; S, $a_2Sa_2^{-1}$, $a_3Sa_3^{-1}$ are pairwise disjoint. We show that $\mathfrak{L}_G$ does not have property Γ. Suppose that a unitary operator, U, in $\mathfrak{L}_G$ is such that $[[UL_{a_j} - L_{a_j}U]] < \epsilon$, for $j=1, 2, 3$, and $0 = T(U) = f(e)$, where $U = L_f$. Define $m(S_0)$, for a subset S_0 of G, to be $\sum_{a\in S_0} |f(a)|^2$, whence $m(G-\{e\}) = 1$, since U is a unitary operator of trace 0. Note that m is additive on G, i.e., with S_2 and S_1 disjoint, $m(S_1 \vee S_2) = m(S_1) + m(S_2)$; so that

$$(*) \qquad m(S) + m(a_2Sa_2^{-1}) + m(a_3Sa_3^{-1}) \leqq m(G - \{e\}) = 1.$$

The inequality condition on U leads easily to

$$|m(S_0) - m(a_jS_0a_j^{-1})| \leqq 2\epsilon, \qquad j = 1, 2, 3,$$

for each subset S_0 of G. Thus

$$(**) \qquad \begin{aligned} 1 = m(G - \{e\}) &= m(S \vee a_1Sa_1^{-1}) \leqq m(S) + m(a_1Sa_1^{-1}) \\ &\leqq 2m(S) + 2\epsilon, \end{aligned}$$

and from (*), $3m(S) - 4\epsilon \leqq 1$, so that

$$\frac{1-2\epsilon}{2} \leqq m(S) \leqq \frac{1+4\epsilon}{3},$$

and $1/14 \leqq \epsilon$. Thus, for no ϵ less than $1/14$, is there a unitary operator satisfying the conditions of property Γ. It remains to note that G_2 has a subset such as S and elements such as a_1, a_2, a_3. In fact, if G is the free product of two groups G_1 and G_2 with G_1 of order not less than 2 and G_2 of order not less than 3; say a in G_1 is distinct from e and e, b, c in G_2 are all distinct, then those words of G which, in reduced form, begin with an element of G_1 form a set such as S with a_1, a_2, a_3 taken as a, b, c, respectively. In particular, the free groups on more than one generator give rise to factors of type II_1 not having property Γ.

In point of fact, Murray and von Neumann do not stop with exhibiting factors such as $\mathcal{L}_{G_p}$ and $\mathcal{L}_{G_2}$, but go on to study a general class of factors of type II_1 of which $\mathcal{L}_{G_p}$ is a particular example. These are the factors $\mathcal{R}$ of type II_1 which are the weak closures of an ascending sequence $\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \cdots$ of factors, $\mathcal{R}_n$, of type I_m. They call such factors "approximately finite" (poor terminology, since it seems to indicate that they are not finite factors), and Dixmier [1] has since replaced this term with the more appropriate "hyperfinite" factors. It comes to the same thing to require that $\mathcal{R}$ be the metric closure of the ascending sequence. In a long series of lemmas involving complicated order of choice arguments, metric approximations, and constructions of subfactors of types I_m with special properties, Murray and von Neumann succeed in showing that all hyperfinite factors are isomorphic. Part of the process entails the establishing of several different criteria for hyperfiniteness of a factor at least one of which, different from the definition, deserves special mention because of its usefulness. If for each finite set of operators $\{A_1, \cdots, A_n\}$ in $\mathcal{R}$ and positive ϵ, one can find a finite-dimensional, self-adjoint subalgebra, $\mathcal{R}_0$, and operators, $B_1, \cdots, B_n$, in it such that $[[A_k - B_k]] < \epsilon$, $k = 1, \cdots, n$, then $\mathcal{R}$ is hyperfinite. It might be thought that a classification, or partial classification of factors of type II_1 can be had, now, by considering factors of type II_1 which are the weak closure of ascending sequences of hyperfinite factors, and so forth; but the criterion just noted implies, at once, that each of these is hyperfinite.

The point to the detailed constructions in the proof of the isomorphism result for hyperfinite factors is to convert the ascending sequence of factors of type I_m, guaranteed by the definition, into a sequence of factors of types I_2, I_4, I_8, $\cdots$. Once this has been accomplished, matching mappings between the subfactors of types I_{2^n} in two distinct hyperfinite factors can be found, without difficulty, and these give rise to an isomorphism between the hyperfinite factors. Beyond these results, the state of knowledge concerning the algebraic nature of factors is very much as Murray and von Neumann left it. In [28], Pukánszky has exhibited two nonisomorphic factors of type III by means of an invariant analogous to property Γ. In both the II_1 and III cases, a third isomorphism class remains to be discovered.

We have not mentioned several topics touched on in the "Rings of Operators" series; normalcy and coupling in factors, the diagonal operation relative to maximal abelian subalgebras, matrix-like representations, principal groups, *-anti-automorphisms. These topics have technical interest, primarily. Two others cannot be passed without further mention, however. In [22], von Neumann initiated, out of technical necessity for the construction of factors of type III, the subject currently called "noncommutative integration theory." In a factor of type II_1, the projections behave like the characteristic functions of measurable sets (with the exception that they do not commute under multiplication), the dimension function as a measure, and the trace as an integration process (on "integrable" operators). The situation, then, is noncommutative in its measure space aspect, rather than with regard to the range of its measure. This theme is fairly well-developed in more general circumstances than just that of a factor of type II_1 by von Neumann who proves, among other things, a Riesz-Fischer result [22, Lemmas 1.4.1–1.4.3]. In [30], Segal carries this theory over to von Neumann algebras in general and also develops the proper general setting for the other topic we shall mention.

Since its inception, the theory of unbounded operators on Hilbert space has held tempting promise for mathematicians who have had contact with it. Elementary formal manipulations of the most reasonable sort pay such high dividends as a solution to the Hilbert fifth problem and difficult results of a purely analytic nature. Indeed, many of the computations of quantum theory are effected by these means. Unfortunately, most of these formal manipulations cannot be justified, and it was von Neumann who pointed out, by example, the pitfalls with which this subject is fraught (cf. Part I of this article). Nonetheless, when such formally appealing maneuvers lead readily to results so much sought after, one cannot help but long for a "world" in which these maneuvers are justified—and von Neumann was, doubtless, no exception. With their discovery of factors of type II_1, Murray and von Neumann succeeded in creating just such a "world." We had occasion to mention unbounded operators affiliated with a von Neumann algebra. Denoting by $[X]$ the closure of an operator X, if such closure exists (i.e., the smallest closed extension of X), in [17, Chapter XVI], Murray and von Neumann prove: that each linear, closed, densely-defined operator affiliated with a factor, $\mathcal{R}$, of type II_1 has no proper closed extension affiliated with $\mathcal{R}$; that such operators, when Hermitian (formally self-adjoint) are self-adjoint; and that if $\mathfrak{U}(\mathcal{R})$ is the set of linear, closed, densely-defined operators affiliated with $\mathcal{R}$ the mappings $X, Y \rightarrow [X+Y]$, $X, Y \rightarrow [XY]$, and $\alpha, X \rightarrow [\alpha X]$ impose the structure of an algebra on $\mathfrak{U}(\mathcal{R})$ closed under the *-operation and relative to which it is a *-algebra in the usual sense.

The crux of the difficulty in formal manipulations with unbounded operators lies in the unrelatedness of the domain and range of one such operator with the domain of another. When we are assured that these sets have many vectors in common, much of the difficulty evaporates. The key to the reasonableness of $\mathcal{U}(\mathcal{R})$, then, is contained in Lemma 16.2.3 of [17] which guarantees just this. Specifically, Murray and von Neumann call a dense linear manifold, M, in $\mathcal{H}$ "essentially dense" when it is the ascending sum of a sequence of closed linear manifolds whose projections belong to $\mathcal{R}$; and they prove that the intersection of a sequence of essentially dense sets is essentially dense. The finiteness of $\mathcal{R}$ is crucial to this argument. It is then proved, in Lemma 16.2.3, that, for each linear, closed, densely-defined operator, X, affiliated with $\mathcal{R}$, the subset of the domain of X consisting of those vectors which map into a given essentially dense set is an essentially dense set. Since $\mathcal{H}$ is essentially dense, it follows that the domain of each such operator is essentially dense. In particular, the intersection of such domains is dense in $\mathcal{H}$. To prove this key result, Murray and von Neumann employ the polar decomposition VH of X, with V a partial isometry in $\mathcal{R}$ and H a positive semi-definite operator affiliated with $\mathcal{R}$. If M is the essentially dense set with $M_1, M_2, \cdots$ an ascending sequence of subspaces associated with it, the inverse image, N_k, of M_k under the bounded operator, VHE_k, is a closed subspace whose projection lies in $\mathcal{R}$—a trivial consequence of the double commutant theorem—where $\{E_\gamma\}$ is the spectral resolution of H. Clearly, $N_1 \subseteq N_2 \subseteq \cdots$, and the inverse image of M under X contains each N_k. The facts that $D(M_k) \to 1$, since the union of the M_k is dense in $\mathcal{H}$ and they form an ascending sequence, and that $D(N_k) \geq D(M_k)$, since a bounded operator in $\mathcal{R}$ maps N_k onto M_k, are easy consequences of dimension theory; and the result follows. From the remark just made, by "weeding" our sequence, we can assume that $D(M_j) > 1 - (1/2^{j+k})$, for any fixed positive k, where $\{M_j\}$ is a sequence associated with some essentially dense set M. If $N_1, N_2, \cdots$ is an infinite sequence of such sets with associated sequences M_{jk} (for N_k) such that $D(M_{jk}) > 1 - (1/2^{j+k})$, then, defining M_j to be $\bigwedge_k M_{jk}$, we have that $M_1 \subseteq M_2 \subseteq \cdots$, $M_j \subseteq \bigwedge_k N_k$, and

$$D(H \ominus M_j) = D\left(\bigvee_k H \ominus M_{jk}\right) \leq \sum_{k=1}^{\infty} \frac{1}{2^{j+k}} = \frac{1}{2^j};$$

so that $\bigvee_j M_j$ is dense in $\mathcal{H}$. Thus, the intersection of an infinite sequence of essentially dense sets is essentially dense.

Two papers, written by von Neumann, [**23**; **24**] are closely related to, though not properly in, the "Rings of Operators" series. In [**23**], von Neumann considers infinite direct products of Hilbert spaces and of the algebras of bounded operators on them. Let Δ be an index set and $\mathcal{H}_\alpha$ a Hilbert space corresponding to each α in Δ. On the space S of all elements $(f_\alpha)_{\alpha\in\Delta}$, in the unrestricted product of the $\mathcal{H}_\alpha$, for which $\Pi_\alpha\|f_\alpha\|$ converges, we consider the space S^* of functionals linear in each coordinate. In S^*, we define S_0^* as the set of finite linear combinations of elements of S^* having the form $\Pi\otimes_\alpha g_\alpha$, where (g_α) is in S and $(\Pi\otimes_\alpha g_\alpha)[(f_\alpha)]=\Pi_\alpha(f_\alpha, g_\alpha)$, for all (f_α) in S. We define the function $\Pi\otimes_\alpha g_\alpha$, $\Pi\otimes_\alpha h_\alpha \rightarrow \Pi_\alpha(g_\alpha, h_\alpha)$, extend it bilinearly, and note that it is an inner product on S_0^*—with which structure S_0^* becomes a pre-Hilbert space. The set of all functionals ϕ in S^* for which there exists a Cauchy convergent sequence (ϕ_n) of elements in S_0^* (relative to the given inner product) such that $\phi[(f_\alpha)]=\lim_n \phi_n[(f_\alpha)]$, for all (f_α) in S, is a Hilbert space—a concrete representation of the "ideal" completion of S_0^*. We denote this space by "$\Pi\otimes_\alpha\mathcal{H}_\alpha$" and refer to it as "the full direct product of the spaces $\mathcal{H}_\alpha$." Each operator, A, on $\mathcal{H}_{\alpha_0}$ gives rise to an operator A_0 on the full direct product, $\mathcal{H}$, determined by $A_0(\Pi\otimes_\alpha f_\alpha)=(\Pi\otimes_{\alpha\neq\alpha_0} f_\alpha)\otimes Af_{\alpha_0}$. The mapping, $A\rightarrow A_0$, is an isomorphism of the algebra, $\mathcal{B}(\mathcal{H}_{\alpha_0})$, of all bounded operators on $\mathcal{H}_{\alpha_0}$ into that, $\mathcal{B}(\mathcal{H})$, of $\mathcal{H}$. At first glance, it might seem that the von Neumann algebra, $\mathcal{B}_0$ generated by all the A_0 (α_0 varying) is $\mathcal{B}(\mathcal{H})$, as is the case for finite direct products (i.e., when Δ is finite). This is not so in general, and, in [**23**, Theorem IX], von Neumann determines the commutant of $\mathcal{B}_0$. It is reasonably clear that each operator U on $\mathcal{H}$ defined by $U[(f_\alpha)]=(a_\alpha f_\alpha)$, where $|a_\alpha|=1$, is a unitary operator which commutes with $\mathcal{B}_0$, and $U=(\Pi_\alpha a_\alpha)I$, provided $\Pi_\alpha a_\alpha$ converges. When $\Pi_\alpha a_\alpha$ does not converge, we have an interesting operator commuting with $\mathcal{B}_0$. Next, let us call an element (f_α) such that $\sum_\alpha \big|\|f_\alpha\|-1\big|$ converges "a C_0-element," and note that each C_0-element lies in S. (The C_0-elements are those elements of S which are well-behaved and nonzero up to elimination of a finite number of coordinate values—and, so, the relevant elements for what follows.) For each C_0-element, (f_α), consider the closed linear space, $\mathcal{H}(f_\alpha)$, in $\mathcal{H}$ generated by those C_0-elements which differ from the given one at no more than a finite number of coordinates. Then distinct $\mathcal{H}(f_\alpha)$ are orthogonal and generate $\mathcal{H}$. Each $\mathcal{H}(f_\alpha)$ is called a "partial direct product" ("incomplete direct product" and the full one "the complete direct product" by von Neumann). In the finite product case, there is one $\mathcal{H}(f_\alpha)$, viz., $\mathcal{H}$, but, in the infinite case, there are more. Clearly, each $\mathcal{H}(f_\alpha)$ is invariant under each A_0, so that the projection, $E(f_\alpha)$, on this space commutes with $\mathcal{B}_0$. The $U(a_\alpha)$, defined above, and the $E(f_\alpha)$ generate the commutant of $\mathcal{B}_0$.

Each operator in $\mathcal{B}_0$ is determined on the space $\mathcal{H}_0$ generated by the images of $\mathcal{H}(f_\alpha)$ under all $U(a_\alpha)$, of course, once it is specified on $\mathcal{H}(f_\alpha)$. Moreover, if we specify bounded operators on each such $\mathcal{H}_0$ consistent with this requirement, the resultant operator on $\mathcal{H}$ will lie in $\mathcal{B}_0$, provided simply that it is bounded (i.e., if and only if the set of bounds of the specified operators is bounded). Thus, the restriction of $\mathcal{B}_0$ to each $\mathcal{H}(f_\alpha)$ is the set of all bounded operators on $\mathcal{H}(f_\alpha)$, i.e., a factor of type I_∞. The really interesting situation occurs when von Neumann considers a special subalgebra of $\mathcal{B}_0$ arising from a particular construction. Let $\{\mathcal{H}_{nm}\}$ be a family of 2-dimensional unitary spaces indexed by pairs of positive integers, $m=1, 2$, and $\mathcal{B}_{nm}$ be its algebra of (bounded) operators. Form $\mathcal{H}=\Pi\otimes_{nm}\mathcal{H}_{nm}$ and $\mathcal{B}_0$ as above. If we denote by $\mathcal{C}$ the von Neumann subalgebra of $\mathcal{B}_0$ generated by operators A_{n1} in $\mathcal{B}_{n1}$ (injected into $\mathcal{B}_0$), $n=1, 2, \cdots$, then $\mathcal{C}$ is just the injection of the $\mathcal{B}_0$ arising from $\Pi\otimes_n\mathcal{H}_{n1}$ into $\mathcal{B}[(\Pi\otimes_n\mathcal{H}_{n1})\otimes(\Pi\otimes_n\mathcal{H}_{n2})]$, and its restriction to a partial direct product is of type I_∞, as before. However, if we form $\mathcal{H}=\Pi\otimes_n(\mathcal{H}_{n1}\otimes\mathcal{H}_{n2})$, inject A_{n1} onto $\mathcal{H}_{n1}\otimes\mathcal{H}_{n2}$ and then onto $\mathcal{H}$ and denote by $\mathcal{C}_0$ the von Neumann algebra generated; the picture changes significantly. To begin with, elementary matrix theory shows that each element g_n of $\mathcal{H}_{n1}\otimes\mathcal{H}_{n2}$ can be put in the normal form

$$g_n = \left(\frac{1+\alpha_n}{2}\right)^{1/2} x_{n1,1}\otimes x_{n2,1} + \left(\frac{1-\alpha_n}{2}\right)^{1/2} x_{n1,2}\otimes x_{n2,2}$$

relative to some bases $x_{n1,1}$, $x_{n1,2}$ for $\mathcal{H}_{n1}$ and $x_{n2,1}$, $x_{n2,2}$ for $\mathcal{H}_{n2}$, where $0\leqq\alpha_n\leqq 1$. Two elements (g_n), (g_n') of $\mathcal{H}$ lie in the same $\mathcal{H}(f_\alpha)$ if and only if

$$\sum_n \left|\frac{1}{2}\left[(1+\alpha_n)^{1/2}(1+\alpha_n')^{1/2}+(1-\alpha_n)^{1/2}(1-\alpha_n')^{1/2}\right]-1\right|$$

converges. In the special case where $1=\alpha_1=\alpha_2=\cdots$, $\mathcal{C}_0$ restricted to the partial direct product, $\mathcal{H}(g_n)$ is of type I_∞. When $0=\alpha_1=\alpha_2=\cdots$, von Neumann establishes that the restriction of $\mathcal{C}_0$ to $\mathcal{H}(g_n)$ is a factor of type II_1. He accomplishes this by constructing an isomorphism with one of the type II_1 examples of a group acting on a measure space (described earlier). More specifically, the space is the unrestricted direct sum of two element groups and the group is the restricted direct sum (a subgroup of the unrestricted sum) acting by translation on the full group. The measure on the space is taken as that induced by Lebesgue measure on the unit interval via the mapping which takes each point $\sum_{m=1}\beta_m/2^m$, $\beta_m=0, 1$, in the unit interval onto the group element (β_m) of the space (this mapping is single-valued and one-one on the complement of the dyadic rationals, a set of Lebesgue measure zero). When α_n is 1 for n even and 0 for n odd, in the construction with 2-dimensional unitary spaces, we are in the case of a product of a factor of type I_∞ with one of type II_1, and one of type II_∞ results. In any event the restriction of $\mathcal{C}_0$ to a partial direct product, in this construction, is a factor, and von Neumann conjectures that none of these are of type III. (Recall that in October of 1937, when [**23**] was submitted, factors of type III had not been constructed.) In [**19**] (cf., Introduction), von Neumann notes that his conjecture was incorrect—in particular, when there exists a positive δ such that $\delta \leqq \alpha_n \leqq 1-\delta$, for infinitely many α_n, the resulting factor is of type III. (A proof of this was never published, though it should not be hard to supply with the techniques of [**22**; **23**].)

The second paper, related to the "Rings of Operators" series which we shall discuss, [**24**], nearly came to the same end as the work on factors of type III mentioned in the introduction to [**19**]. In fact [**24**] is alluded to in the introduction to [**23**] but appears in 1949 (at the request of Mautner who needed its results for some of his work on group representations [**16**]). The substance of [**24**] is the description of a process by which a direct integral of Hilbert spaces and von Neumann algebras thereon, over a measure space, can be formed. If the measure space is totally atomic the direct integral reduces to a direct sum, and the general process bears the same relation to forming the direct sum that general integration theory does to discrete summation. The inverse process is also a major concern of [**24**]—i.e., the process of "reducing" a Hilbert space and von Neumann algebra thereon to a direct integral of Hilbert spaces and von Neumann algebras on each, relative to an abelian von Neumann algebra in the commutant of the original von Neumann algebra. One of the principal results of the paper (and, perhaps, its primary motivation) is the theorem that each von Neumann algebra can be reduced, relative to its center, to a direct integral of factors (the Hilbert space constructs are unique up to sets of measure 0 and the measure up to absolute bicontinuity). This result gives some justification for the preponderance of attention paid by Murray and von Neumann to factors over the general von Neumann algebra and supplies a possible technique for analyzing von Neumann algebras in terms of factors. In point of fact, however, the experience of the past decade has shown us that, when specific reduction results are not in question, it is best to deal with von Neumann algebras by global techniques thereby avoiding intricate measure-theoretic difficulties.

Let X be a locally compact space and m a positive measure on X. A mapping, $x \to \mathcal{H}(x)$, associating with each point of X a Hilbert space is said to be "an m-measurable field of Hilbert spaces" when there exists a linear subspace L of the product $\prod_{x \in X} \mathcal{H}(x)$ $(=P)$ such that

(a) for a in L, $x \to \|a(x)\|$ is measurable;

(b) if $b \in P$ and $(b(x), a(x))$ is measurable, for each a in L then b is in L;

(c) there is a denumerable set $\{a_n\}_{n=1,2,\ldots}$ of elements of L such that the closed subspace generated by $\{a_n(x)\}$ is $\mathcal{H}(x)$, for each x in X.

The set, $\mathcal{H}$, of elements, a, in P for which

$$\int_X \|a(x)\|^2 dm(x) < \infty$$

forms a Hilbert space relative to the inner product

$$(a, b) = \int_X a(x)\bar{b}(x) dm(x),$$

and this Hilbert space $\mathcal{H}$ is called "the direct integral (written: $\int_X \oplus \mathcal{H}(x) dm(x)$) of the $\mathcal{H}(x)$ over X, relative to m." If the mapping, $x \to T(x)$ (an operator on $\mathcal{H}(x)$) is such that $x \to \|T(x)\|$ is measurable and essentially bounded, the operator T on $\mathcal{H}$ defined by $(T(a))(x) = T(x)a(x)$ is bounded with bound equal to the essential supremum of $x \to \|T(x)\|$. Operators such as T on $\mathcal{H}$ are said to be "decomposable." These definitions and results provide the background for the direct integral and reduction theory described in the preceding paragraph.

A problem of some importance, for a complete understanding of the structure of von Neumann algebras, is that of characterizing them abstractly (independently of a concrete representation on a Hilbert space). In [26], von Neumann takes up this question. The appropriate techniques for this problem were not at hand at that time, and, while some progress was made, a satisfactory solution was not obtained. Gelfand and Neumark succeeded in finding an excellent characterization of a broader class of self-adjoint operator algebras than the von Neumann algebras, the so-called C^*-algebras (those closed in the uniform topology on operators) [6]. As currently revised, their result states: each Banach algebra with a conjugate-linear, anti-automorphic, involutory*-operation (like the adjoint operation for operators on a Hilbert space) for which $\|a^*a\| = \|a^*\| \cdot \|a\|$ is *-isomorphic with a uniformly closed self-adjoint operator algebra. (The mapping is isometric on elements which commute with their own adjoint and regular elements—and, very likely on all elements, though this is not known. With the additional assumption, $\|a\| = \|a^*\|$, the isomorphism can be shown to be isometric.) Following the results of [6], Rickart and Kaplansky [**14**; **15**; **29**] developed an "algebraic" theory of von Neumann algebras in a subclass of the C^*-algebras which had the formal algebraic properties of the von Neumann algebras (but which was known to be a properly larger class). Many of the global techniques, which make the current use of von Neumann algebras a powerful and effective tool, stem from these algebraic investigations. From the Gelfand-Neumark result, our algebraic characterization question becomes: when is a C^*-algebra of operators on a Hilbert space *-isomorphic with a von Neumann algebra? The condition that each bounded set of self-adjoint operators in the algebra, directed by the natural ordering on such operators, has a least upper bound in the algebra is the crucial algebraic feature of von Neumann algebras as distinguished from the general C^*-algebra. (Conditions related to this are used in [**14**; **29**].) To answer the characterization question, we assume this condition holds for our C^*-algebra. Such an algebra is *-isomorphic with a von Neumann algebra if and only if there exists a separating family of states whose limits on a directed, bounded family of self-adjoint operators is their value at the limit [**12**] (this is equivalent to the states being normal).

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