BOOK REVIEWS

An introduction to Diophantine approximation. By J. W. S. Cassels. Cambridge Tracts in Mathematics and Mathematical Physics, no. 35, New York, Cambridge University Press, 1957. 10+166 pp. $4.00.

The theory of Diophantine approximation is concerned with the approximate solution of Diophantine equations having no exact solutions, or only trivial ones. The first such problem considered was probably that of finding the good rational approximations to a real irrational number θ, or what is almost the same thing, of finding integers x, y, not both zero, for which $|\theta x-y|$ is small. This problem was generalized in the early 1840's by Dirichlet, who considered $|\theta_1x_1+\cdots+\theta_nx_n-y|$, and about 1850 by Hermite, who considered the system of quantities $|\theta_1x-y_1|, \cdots, |\theta_nx-y_n|$. These papers, together with that of Liouville (1844) on the existence of transcendental numbers, might be regarded as the beginnings of the subject. Much later, Minkowski suggested the term "Diophantine approximation," and used it as title for the first book (1907) in the subject.

In 1936 J. F. Koksma's *Diophantische Approximationen* appeared in the Ergebnisse series. This beautiful work describes, usually without proofs, all the important research up to the time of publication, and contains an exhaustive bibliography. It is badly out of date now, of course, because the subject is one of the most active branches of number theory. A present day bibliography would probably contain at least twice as many references as the approximately 900 in Koksma's book. Strangely enough, the additional references would be nearly exclusively to papers by European and Russian authors; it is almost as if the subject were nonexistent as far as American mathematics is concerned.

Since Koksma's book, no general work has appeared until the tract under review, a fact which further increases the value of what is, in its own right, a very important piece of work. Cassels' book does not have the scope of Koksma's, and indeed it could not have under the space limitations imposed in the Cambridge series. No attempt is made to present a complete bibliography, nor to discuss all the important problems and topics in the field. (For example, just one paragraph is devoted to transcendental numbers.) Rather, the author presents a relatively small number of theorems, each with complete proof. Many of the most beautiful and significant results of the past 20 years are to be found, in most cases with new or simplified proofs appearing here for the first time. Indeed, of the eight chapters only that on Roth's theorem is a rewrite of published work. The intending reader should be warned, however, that not all the economy of space comes about through genuine simplification; it is partly achieved by omissions of discussion, description and historical remarks, and by a terse style in the proofs themselves. No doubt space restrictions by the publishers made this necessary, and it is unquestionably better to have such excellent subject matter presented in somewhat condensed form than not at all. In any case, the book is not easy reading, and a beginner would be well advised to have pencil and paper at hand.

Chapter I, on homogeneous approximation, is concerned with systems of linear forms of the type mentioned above in connection with Dirichlet. In the simple case of a single form $|\theta x - y|$, there is available the immensely powerful theory of regular continued fractions, the usefulness of which derives from the fact that the convergents of the continued fraction expansion of θ are precisely the best approximations to θ. By a best approximation is meant a rational number p/q ($q>0$) such that $|q\theta - p| < \|r\theta\|$ for all integers r with $0<r<q$. (Here $\|x\|$ represents the distance between x and the integer nearest x, so that $\|x\| = \min |x-n|$, the minimum extending over all integers n.) The author reverses the usual procedure; instead of defining a continued fraction and then verifying that its convergents have the desired property, he constructs the continued fraction as the solution of the problem of finding the best approximations. This is advantageous for generalization, as it points the way to the construction of the best approximations (from among the elements of a subfield) to an element of an arbitrary field of complex numbers, and it puts the emphasis on that aspect of continued fractions which is of greatest importance in number theory.

The remainder of the first chapter is devoted to the Dirichlet-Minkowski theorem, and a simple proof that that theorem is in a sense best possible.

Chapter II deals with the Markoff theory of the minima of indefinite binary quadratic forms. The proof presented goes back to works by Frobenius, Remak, C. A. Rogers and the author. Chapter III is concerned with inhomogeneous linear approximation, including Minkowski's theorem on the product of two linear forms, and Kronecker's theorem.

In Chapter IV on uniform distribution there is given a simple proof, from the definition, that the sequence $\{n\theta\}$ is uniformly distributed (mod 1) for irrational θ; the proof clearly generalizes to the case of n-tuples of linear forms in m integral variables, if no nontrivial integral linear combination of the forms has integral coefficients. Weyl's criterion is then developed, and used to prove that if a polynomial $f(x)=a_rx^r+\cdots+a_0$ has at least one irrational coefficient a_j with $j>0$, then the sequence $\{f(n)\}$ is uniformly distributed (mod 1).

Chapter V contains a collection of important transference theorems (Übertragungssätze), by means of which information about one set of forms yields information about another set. The prototype of this class of theorems is due to Perron and Khintchine: Let $\theta_1, \cdots, \theta_n$ be irrational numbers, and let ω_1 and ω_2 be the respective upper bounds of the numbers ω, ω' such that the inequalities

$$\|u_1\theta_1+\cdots+u_n\theta_n\| \leqq (\max |u_j|)^{-n-\omega},$$
$$\max_{1\leqq j\leqq n} \|x\theta_j\| \leqq x^{-(1+\omega')/n}$$

have infinitely many integer solutions $(u_1, \cdots, u_n)$ and x. (ω_1 and ω_2 are nonnegative by the Dirichlet-Minkowski theorem.) Then

$$\omega_1 \geqq \omega_2 \geqq \frac{\omega_1}{n^2+(n-1)\omega_1}\cdot$$

More generally, the homogeneous approximation of a set of forms $\sum_i \theta_{ji}x_i$ can be related to that of the transposed set $\sum_j \theta_{ji}u_j$, and also to the inhomogeneous approximation of each of the sets. These theorems and the techniques involved in their proofs are used in the further consideration of some of the material of Chapter III.

Roth's remarkable improvement of the Thue-Siegel theorem is presented in Chapter VI. This theorem asserts that if ξ is an irrational algebraic number and δ is positive, then the inequality

$$\left|\xi-\frac{p}{q}\right|<\frac{1}{q^{2+\delta}}$$

has only finitely many solutions in integers $q>0$, p. The proof is succinct as usual, having shrunk to fifteen pages from Roth's original twenty.

Chapter VII is concerned with "almost all" type results. The author restricts himself to the question of the solvability of the inequalities $\|q\theta_j-\alpha_j\|<\psi(q)$, $1\leqq j\leqq n$. The easier result is that if $0\leqq\psi(q)\leqq 1/2$ for all q, then these inequalities have infinitely many solutions for almost no or almost all $2n$-dimensional sets $(\theta_1, \cdots, \theta_n, \alpha_1, \cdots, \alpha_n)$, according as the series $\sum(\psi(q))^n$ converges or diverges. It is also shown, with considerably more difficulty, that in the case $\alpha_1 = \cdots = \alpha_n = 0$, the additional hypothesis that ψ is monotone decreasing guarantees that the inequalities in question have infinitely many solutions for almost no or almost all sets $(\theta_1, \cdots, \theta_n)$, according as the above series converges or diverges. This hypothesis is weaker than that in the similar theorem of Khintchine.

In the final chapter the Pisot-Vijayaraghavan (PV) numbers are studied. These are the algebraic integers $\alpha > 1$ all of whose conjugates except α itself lie in the open disk $|z| < 1$. It is easy to see, by considering the trace of α^n, that $\|\alpha^n\|$ approaches zero as $n \to \infty$ if α is a PV number. Pisot showed, conversely, that if $\alpha > 1$ is algebraic and $\lambda \neq 0$ is real, and if $\|\lambda\alpha^n\| \to 0$, then α is a PV number. Moreover, he showed that if $\alpha > 1$ is real, and if $\sum \|\lambda\alpha^n\|^2 < \infty$, then α is algebraic and therefore a PV number. (It is an open question whether $\|\lambda\alpha^n\| \to 0$ implies that α is algebraic.) Salem obtained the unexpected result that the set of PV numbers is closed. Proofs are given here of these three theorems; they are much simpler than the original proofs, although similar in conception.

The book closes with three appendices giving necessary tools from linear algebra and geometry of numbers, and a bibliography of papers mentioned.

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Irrational numbers. By Ivan Niven. Carus Monograph no. 11: New York, Wiley, 1956. xii+164. $3.00.

This most recent in a series of distinguished monographs is outstanding in organization, in clarity, and in choice of material. The book, which begins with "the preponderance of irrationals," and which closes, in Chapter X, with a proof of the Gelfond-Schneider theorem, is an admirable fulfillment of the author's purpose: "an exposition of some central results on irrational numbers . . . the main emphasis [being] on those aspects . . . commonly associated with number theory and Diophantine approximations."

The topics are arranged, in general, in order of difficulty, with the result that some of the theorems in the early part of the book are subsumed under stronger theorems later. This organization seems to have real pedagogical value. The same sort of organization is followed, to some extent, within each chapter; for example, a theorem on the uniform distribution of a sequence of irrationals is first proved by use of results on continued fractions (one of the few cases where appeal is made to the material of an earlier chapter), and then ob-