

THE BOUNDARY OF A SIMPLY CONNECTED DOMAIN¹

GEORGE PIRANIAN

1. **Prime ends.** The foundation for the study of boundaries of simply connected domains in the plane was laid by Carathéodory [2], who defined ends in general and prime ends in particular, and who classified prime ends into four kinds. To facilitate a brief survey of the subject of prime ends, I introduce a few definitions concerning a simply connected domain B . In my definitions, I follow essentially the work of Carathéodory, except that, for the sake of brevity, I omit the description of ends in general and aim directly at prime ends.

A sequence K of crosscuts c_n of B is a *chain* provided

- (i) the diameter of c_n tends to 0 as $n \rightarrow \infty$;
- (ii) for each index n , the set $\bar{c}_n \cap \bar{c}_{n+1}$ (where the bar indicates closure) is empty;
- (iii) some fixed point O in B cannot be joined to any crosscut c_n ($n > 1$) by any path in B which does not meet the crosscut c_{n-1} .

Two chains $K = \{c_n\}$ and $K' = \{c'_n\}$ in B are *equivalent* provided each crosscut c_n effects a separation, relative to B , of the point O from all except finitely many of the crosscuts c'_n .

An equivalence class of chains in B is a *prime end* of B .

If P is a prime end of B , let $K = \{c_n\}$ denote a chain which belongs to P , and for each index n let B_n denote that subdomain of B which is determined by c_n and does not contain the point O . The set $I(P) = \bigcap \bar{B}_n$ will be called the *impression* of P .

It should be remarked that Carathéodory and some other writers applied the term *prime end* to the point set $I(P)$, but that they regarded as distinct two prime ends P_1 and P_2 corresponding to two nonequivalent chains, even in cases where the two sets $I(P_1)$ and $I(P_2)$ are identical. The distinction between a prime end and its impression formalizes the ideas which are involved.

Carathéodory's principal theorem on the correspondence between boundaries under conformal mappings [2, p. 350] can be expressed as follows: If $f(z)$ maps the unit disk conformally and one-to-one onto the

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domain B , it induces a one-to-one mapping between the points of the unit circle and the prime ends of B .

2. Alternate definitions, and extensions. Koebe [10, pp. 217–219] introduced an alternate definition of prime ends (boundary elements). It is based on equivalence classes of paths to accessible boundary points. Two boundary paths Γ_1 and Γ_2 in B (that is, two homeomorphic images $p_i = p_i(t)$ ($i=1, 2$) of the open unit interval) are equivalent if they both approach the same boundary point $p(1)$ as $t \rightarrow 1$ and if, for each $\epsilon > 0$, every point $p_1(t)$ ($t > t_\epsilon$) can be joined to Γ_2 by a path which lies in B and has diameter less than ϵ . Prime ends correspond to “Dedekind sections” in the space of equivalence classes of paths to accessible boundary points.

Ursell and Young [16, see (4.5) and (4.6) on p. 5] took an approach intermediate between that of Carathéodory and that of Koebe. Their principal contributions are the study of the fine structure of the individual prime end, and the investigation of the relation between prime ends of B and the interior of the complement of B .

It is natural that the concept of prime ends should be subjected to axiomatization, and that it should be extended to domains of infinite connectivity, to domains in euclidean three-space, and finally to abstract spaces. In this direction, papers by Kaufmann [9], Mazurkiewicz [13], and Freudenthal [6] must be mentioned. Suvorov [15] defined prime ends of a sequence of domains converging to a nucleus. But these are developments which lead away from my problem, and I return to the ideas of Carathéodory.

3. The classification of prime ends. A point p in the impression $I(P)$ is a *principal point* (relative to P) provided every neighborhood of p contains a crosscut of a chain which belongs to P ; otherwise, it is a *subsidiary point* (relative to P). Elementary considerations show that every impression $I(P)$ contains at least one point which is a principal point relative to P .

A prime end is of the *first kind* if its impression consists of a single point, which is then necessarily a principal point. In other words, P is of the first kind provided

- (a) $I(P)$ contains only one principal point, and
- (b) $I(P)$ contains no subsidiary points.

A prime end is of the *second, third, or fourth* kind if it satisfies only condition (a), only condition (b), or neither of the two conditions, respectively.

For the sake of brevity, we shall also say that a point set on the boundary of B is an impression of the first, second, third, or fourth

kind if it is the impression of a prime end of the corresponding kind.

By way of illustration, we consider the domain suggested by Figure 1. The domain consists of the interior of a square from which various line segments have been deleted. Each impression on the upper edge AC of the square is of the first kind; the same is true of impressions lying on slits that issue from the upper edge. The point B is the impression of infinitely many prime ends.

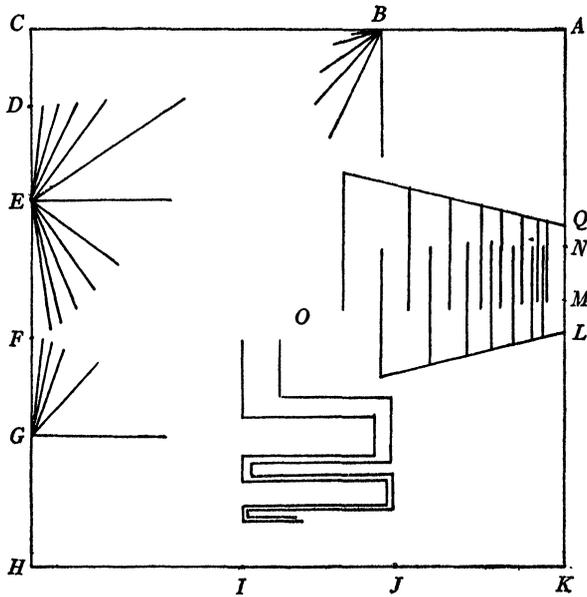


FIG. 1

Each impression on the side CH or on one of the slits issuing from CH is of the first or second kind. In particular, there are two impressions of the second kind, namely the segments DE (with D as principal point) and EG (with F as principal point). The point sets consisting of E and G , respectively, are the impressions of infinitely many prime ends of the first kind.

On HK , there are two impressions of the second kind, namely the segment IJ with I as principal point, and the same segment with J as principal point. In addition, the segment IJ is an impression of the third kind. If the figure is modified by adjoining infinitely many further slits (following closely the slits that approach the segment IJ), then IJ becomes the impression of infinitely many prime ends of the third kind.

On KA and the adjoining slits, all impressions are of the first kind, except that the segment LQ is an impression of the fourth kind. The points on the closed segment MN are principal points; the points on the remainder of the closed segment LQ are subsidiary points. The two point sets consisting of L and Q , respectively, are impressions of the first kind.

4. The distribution problem. Urysohn observed [17, p. 235] that the one-to-one correspondence between the unit circle and the set of prime ends of a simply connected domain B induces a Hausdorff topology in the set of prime ends of B . We denote by $U(B)$ the topological space of prime ends of B , and by $U_k = U_k(B)$ ($k=1, 2, 3, 4$) the set of prime ends in $U(B)$ which are of the k th kind. A question of Carathéodory [2, p. 325] can now be phrased as follows: Is it possible that $U_2(B) = U(B)$?

This question was answered in the negative by Weniainoff [18], who applied Lindelöf's theorem to characterize the prime ends of the first and second kinds in terms of the radial behavior of the mapping function $f(z)$ which carries the unit disk onto B . The function $f(z)$ carries a point $e^{i\theta}$ into a prime end of the first or second kind if and only if the radial limit $f(e^{i\theta})$ of $f(z)$ exists. This implies that if the sets $U_3(B)$ and $U_4(B)$ are empty, the function $f(e^{i\theta})$ exists everywhere. Also, $f(e^{i\theta})$ is then of Baire class 1, and its set of discontinuities is therefore of first category. Finally, the set $U_2(B)$ is the image of the set of discontinuities of $f(e^{i\theta})$, and therefore $U_2(B)$ is also of first category. Essentially the same argument was used, independently, by van Aardenne-Ehrenfest and Wolff [1]. Another solution of the Carathéodory problem is due to Urysohn [17].

The argument of Weniainoff actually yields more than I have stated so far: *If the sets U_3 and U_4 are empty, then U_2 is of first category and of type F_σ .* Lohwater and I have proved a converse of this [12]: *Let E_2 be any set on the unit circle C , of type F_σ and of first category. Then there exists a function $f(z)$, bounded and schlicht in D , which carries each point of E_2 into a prime end of the second kind, and each point of $C - E_2$ into a prime end of the first kind.*

The fact that Carathéodory's special question on the set U_2 is amenable to such complete treatment constitutes a temptation to propose the following problem.

PROBLEM 1. *To find necessary and sufficient conditions on the decomposition of the unit circle C into four disjoint sets E_k ($k=1, 2, 3, 4$) in order that, under some conformal mapping of the unit disk onto a bounded schlicht domain, the points of each set E_k correspond to prime ends of the k th kind.*

The solution of this general problem will be vastly more difficult than the solution of the special problem where U_3 and U_4 are empty. The difference is not merely one of degree, but it goes very deep; for the solution of the special problem can be stated in terms of topological concepts alone, while Fatou's theorem [5, p. 337] implies that the point sets E_3 and E_4 are subject to metric restrictions. Pending the development of methods and skills which are not available today, it seems advisable to replace Problem 1 by a less forbidding version:

PROBLEM 2. *To find necessary and sufficient conditions on the decomposition of C into four disjoint sets E_k ($k=1, 2, 3, 4$) in order that, for some simply connected domain B and some appropriate homeomorphism between C and $U(B)$, each set E_k corresponds to the set $U_k(B)$.*

5. Necessary conditions on the decomposition. Throughout this section, B denotes a fixed (but arbitrary) simply connected domain; P denotes a prime end of B , and $|P|$ the diameter of P , defined as the diameter of its impression $I(P)$.

THEOREM 1. *For every positive number h , the set of prime ends of diameter at least h is closed.*

COROLLARY. *The set $U_1(B)$ is of type G_δ .*

PROOF. Let $\{B_n\}$ be a chain of domains corresponding to P , and suppose that every neighborhood of P (in the sense of Urysohn) contains a prime end of diameter at least h . Then every set \bar{B}_n contains a subset of diameter at least h , and therefore $|P| \geq h$. The corollary follows since $U_1(B) = \bigcap M_n$, where M_n denotes the set of prime ends of diameter less than $1/n$ ($n=1, 2, \dots$).

For the further analysis, we need the concept of cluster sets. A complex number w belongs to the (complete) *cluster set* of the function $f(z)$ at $e^{i\theta}$ provided there exists a sequence of points z_n in the unit disk D , converging to $e^{i\theta}$, and having the property that $f(z_n) \rightarrow w$ as $n \rightarrow \infty$. The point w belongs to the *radial cluster set* of $f(z)$ at $e^{i\theta}$, if the points z_n can be chosen on the radius of $e^{i\theta}$.

If f maps the unit disk D conformally onto B in such a way that the center of D is carried to the point O in B , and if $e^{i\theta}$ is the point corresponding to the prime end P with the chain $\{c_n\}$, then the image of the radius of $e^{i\theta}$ meets each of the cuts c_n . It follows that every principal point of P belongs to the radial cluster set of f at $e^{i\theta}$. On the other hand, Lindelöf has shown [11, p. 28] (see also Montel [14, pp. 48–52] and Gross [7, p. 254]) that the radial cluster set at $e^{i\theta}$ contains no subsidiary points of P . It follows that all points $e^{i\theta}$

at which the radial limit exists correspond to prime ends of the first or second kind, and that all points $e^{i\theta}$ at which the radial cluster set coincides with the complete cluster set correspond to prime ends of the first or third kind.

THEOREM 2. *On every interval of $U(B)$, the set $U_1 \cup U_2$ has the power of the continuum.*

This proposition follows immediately from Fatou's theorem. It can also be proved directly by showing that every interval of impressions of prime ends contains 2^{\aleph_0} points that are appropriately accessible from the interior of B .

THEOREM 3. *The set $U_3 \cup U_4$ is of type $G_{\delta\sigma}$.*

The theorem follows from the fact that the set of radii on which f has positive oscillation is a set of type $G_{\delta\sigma}$ (see Hausdorff [8, p. 273]).

Before closing this section, we call attention to a recent result of Collingwood [3] which implies that if the function f is meromorphic in D , then there exists a residual set E on C such that, at each point $e^{i\theta}$ in E , the radial cluster set at $e^{i\theta}$ coincides with the complete cluster set at $e^{i\theta}$. It follows (see Collingwood [4, p. 349]) that for every simply connected domain B the set $U_1 \cup U_3$ is residual.

6. The case where U_4 is empty. The results of the preceding section, together with an appropriate geometrical construction, lead to the following partial solution of the second distribution problem.

THEOREM 4. *Let B be a simply connected domain without prime ends of the fourth kind. Then there exist two sequences of sets F_n and M_n in the space $U(B)$ which have the following three properties.*

1. *The sets F_n are mutually disjoint, and the set $\bigcup_{n=1}^m F_n$ is closed, for every m .*

2. *For each n , the set M_n is of type G_δ and is contained in F_n ; also, on every open subset of F_n , both M_n and its complement have the power of the continuum.*

3. $U_1 = U - \bigcup F_n$, $U_2 = \bigcup (F_n - M_n)$, $U_3 = \bigcup M_n$.

On the other hand, let $\{F_n\}$ and $\{M_n\}$ be two sequences of sets on C such that Conditions 1 and 2 are satisfied, and let

$$E_1 = C - \bigcup F_n, \quad E_2 = \bigcup (F_n - M_n), \quad E_3 = \bigcup M_n.$$

Then there exists a domain B such that some homeomorphism between C and $U(B)$ carries each of the sets E_k into the corresponding set $U_k(B)$ ($k = 1, 2, 3$).

To prove the first part of the theorem, we let F_1 denote the set of

prime ends P with $|P| \geq 1$, and for $n=2, 3, \dots$ we let F_n denote the set of prime ends P with $1/(n-1) > |P| \geq 1/n$. Condition 1 and the first part of Condition 3 are then satisfied, by Theorem 1.

Let f denote a function which maps the unit disk D conformally onto B , and let E_{n_j} be the set of points $e^{i\theta}$ for which there exist two values r_1 and r_2 ($1-2^{-j} < r_1 < r_2 < 1$) such that

$$|f(r_1 e^{i\theta}) - f(r_2 e^{i\theta})| > 1/n - 1/2^j.$$

Then E_{n_j} is open, and therefore the set $E_n = \bigcap_{j=1}^{\infty} E_{n_j}$ is of type G_δ . Now let M_n^* be the set of prime ends of B which correspond to points in E_n . Then each prime end in M_n^* lies in $\bigcup_{m=1}^n F_m$, by virtue of its diameter. Since E_n consists of all points of C on whose radius the function f has oscillation of amplitude at least $1/n$, and since U_4 is empty (by hypothesis), M_n^* consists of all prime ends of the third kind and of diameter at least $1/n$.

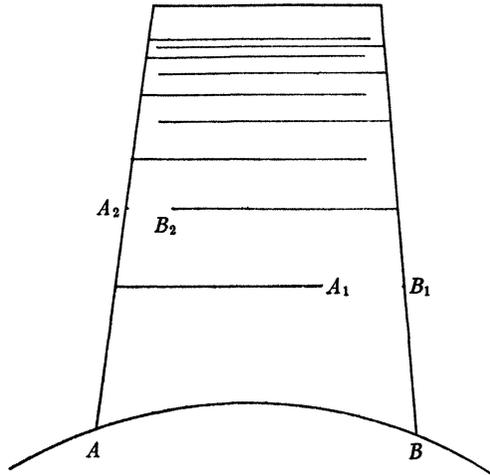


FIG. 2

Let $M_n = M_n^* \cap F_n$. The first part of Condition 2 is clearly satisfied; the second part follows from the theorem of Weniaminoff and the fact that a dense set of type G_δ is not denumerable.

7. The Carathéodory boxes. To complete the proof of the theorem, we use a device which is a slight modification of a construction described by Carathéodory [2; see the figure on p. 366 and the discussion on p. 369].

Let $\{F_n\}$ and $\{M_n\}$ be two sequences of point sets on C , subject to Conditions 1 and 2 of Theorem 4. Then each set M_n can be repre-

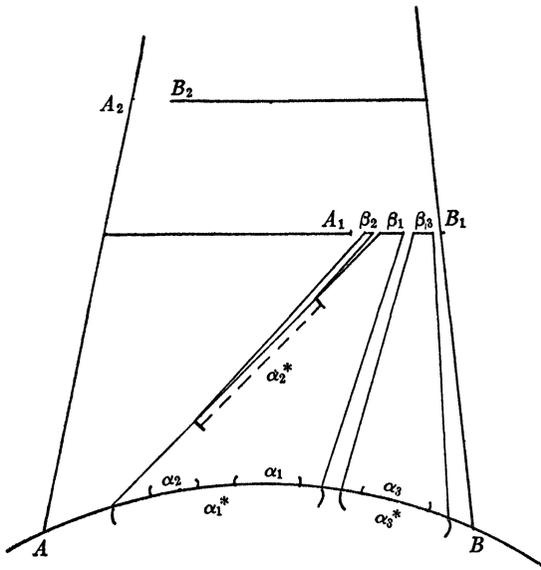


FIG. 3

sented as the intersection of a decreasing sequence of open sets G_{nk} . We confine our attention to a single one of the open arcs AB constituting the set G_{11} . With this arc as base we construct a "trapezoid" lying outside of C (see Figure 2). Inside of the trapezoid we draw infinitely many rectilinear crosscuts, each parallel to the base opposite AB , and with the distance between the j th crosscut and that base tending to 0 decreasingly, as $j \rightarrow \infty$. From the j th crosscut we delete a short segment A_jB_j at the right or left end, according as j is odd or even. And we imagine the arc AB mapped onto each of the deleted segments A_jB_j , by any homeomorphism which carries the point A into A_j .

By Condition 2 in the theorem, there exists in the closure of AB a perfect set which lies in the complement of M_1 and which contains both the points A and B . We choose such a set and denote it by N_0 . Without loss of generality, we may assume that each of the open arcs of AB contiguous to N_0 contains at least one point of M_1 ; for any arc which fails in this respect can simply be incorporated into N_0 .

Let $\alpha_1, \alpha_2, \dots$ denote the open arcs that compose the set $AB - N_0$. We erect a trapezoid which has for one side the image β_1 of α_1 on A_1B_1 , and for the opposite side an arc α_1^* on AB which contains α_1 and whose end points are essential points of N_0 (see Figure 3; by an essential point of N_0 we mean a point of N_0 which is not an

end point of any arc of the complement of N_0). We imagine the two segments of α_1^* which are complementary to α_1 mapped onto the corresponding sides of the quadrilateral which has just been erected.

If the arc α_2 does not lie on α_1 , we accord it similar treatment. If it lies on α_1 , we give it a quadrilateral one of whose sides is the image β_2 of α_2 on A_1B_1 , while the opposite side is a segment α_2^* on the corresponding side of the quadrilateral over α_1 ; the segment α_2^* must contain the image of α_2 , and its endpoints must be images of essential points of N_0 .

We continue thus, with the proviso that no segment α_k^* shall have in its interior or as one of its endpoints any vertex of a previously constructed quadrilateral.

After each of the arcs α_k has been treated in this manner, we delete the arcs α_k^* . Now the unit disk, together with certain extensions that reach to the segment A_1B_1 , constitutes a simply connected domain. And the open subset $\cup \alpha_k$ of G_{11} , which contains the set $M_1 \cap AB$, has been mapped onto the open set $\cup \beta_k$ on A_1B_1 .

Next we consider the intersection of this open subset of A_1B_1 with the image of $G_{12} \cap AB$ on A_1B_1 . From the closure of each component of this intersection we delete a perfect set N_1 which contains both endpoints of the component and whose pre-image does not meet the set M_1 . We then repeat our construction with the open arcs $\alpha_k^{(1)}$ that remain; that is, for each of these open arcs we construct a quadrilateral one of whose sides is the image $\beta_k^{(1)}$ on A_2B_2 of the arc.

The result of the continuation of the process is called a *Carathéodory box*. It is easily seen that the unit disk together with the Carathéodory box over AB is a simply connected domain.

Consider now any chain $K = \{c_n\}$ of crosscuts which have at least one end in the Carathéodory box. Either the crosscuts c_n approach the upper edge, that is, the lid of the box, in which case K belongs to a prime end P of the third kind, and $I(P)$ coincides with the lid; or the crosscuts approach a point which is not on the lid. To analyze the latter case, we observe that every quadrangle which is erected during the construction is ultimately supplied with extensions which reach to the lid of the box; that every point which is a vertex of a quadrilateral is a limit point of such vertices; and that every vertex of a quadrilateral is a boundary point of the domain. If a chain of crosscuts approaches a point p which is a limit point of vertices of quadrilaterals and does not lie on the lid, then the chain belongs to a prime end P of the second kind; p is the principal point of P ; and $I(p)$ consists of an infinite "polygonal line" and of the lid of the box.

It follows that there exists a homeomorphism between the unit

circle and the space of prime ends of our domain which maps into prime ends of the third kind all points on $M_1 \cap AB$; into prime ends of the second kind all limit points of this set which do not belong to the set; and into prime ends of the first kind all the remaining points of the circle.

It is easily seen how the same process can be applied to all other arcs of G_{11} . If the continuation is effected with uniformly high Carathéodory boxes, then the result is a domain which has prime ends of the third kind corresponding to all points of M_1 , and prime ends of the second kind corresponding to all points of $\overline{M_1} - M_1$.

Meanwhile, the open circular arcs of $C - \overline{M_1}$ have been transformed into open continuous arcs. On these arcs we can erect "parallel quadrilaterals" (in the manner of [12, §2]) in such a way that all points of $F_1 - M_1$ correspond to prime ends of the second kind.

The set $C - F_1$ is open, and its image consists of open continuous arcs. On these arcs we construct Carathéodory boxes corresponding to the set M_2 , and further quadrilateral extensions corresponding to the set $F_2 - \overline{M_2}$. The Carathéodory boxes and the extensions may be distorted; but they can be made uniformly large, except near the impressions of prime ends corresponding to points of F_1 . The indefinite continuation of the process presents no difficulty; for after the completion of each stage, all further modifications have to be made on open, continuous arcs of the boundary. And while the process would be certain to drive any draftsman to despair, it yields a domain B which constitutes a proof of the second part of Theorem 4.

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