BOOK REVIEWS

Mathematical logic. By R. L. Goodstein. Leicester, Leicester University Press, 1957. 8+104 pp. 21s.

In the preface, the author states that his aim is "to introduce teachers of mathematics to some of the remarkable results . . . in mathematical logic during the past twenty-five years." He goes on to say that the book is designed for mathematicians with little or no previous knowledge of symbolic logic and is largely self contained in that proofs of major results are given in detail. He concludes "a great many different facets of the subject have been briefly sketched, but rigour has not been intentionally sacrificed for ease of reading."

This short book covers a wide range of topics. In the Introduction (10 pp.) the Frege-Russell definition of number serves both as an illustration of a representative concern of logic and as an opportunity for introducing the reader to customary logical notations. The necessity of the class concept in treating cardinal numbers is indicated. The formalistic calculus-of-numerals approach to number theory is also discussed, and the observation is made that it is different in level rather than correctness.

Chapter I (17 pp.) on *sentence calculus* begins with truth-table validity. The author's use of numerical representing functions at this point is engaging and instructive. The Russell-Whitehead axiomatization is then described and a deduction theorem outlined. Completeness (not proved), consistency, independence (proved for one axiom), three-valued logic, bracket-free notation (with Łukasiewicz's axioms) are discussed, a natural inference formulation is set forth and asserted to be complete. The Heyting intuitionist system is given.

Chapter II (16 pp.) on (lower) *predicate calculus*, gives both an axiomatic formulation with deduction theorem and Quine's natural inference formulation. Validity, satisfiability, and decision for finite domains and monadic calculus are presented. The Gödel completeness theorem is reached through Henkin's proof of the Skolem-Löwenheim theorem.

Chapter III (29 pp.) on *number theory* presents first the Hilbert-Bernays system Z and then R. Robinson's finitely axiomatizable subsystem Z_f. The theory of primitive recursive functions is briefly developed and use of the Chinese remainder theorem in showing primitive recursive functions to be arithmetical is outlined. (Numeralwise expressibility is stated but not proved.) More general kinds of recursion are described, and reductions to primitive recursion from course-of-values, parameter and simultaneous recursion are shown. General ordinal recursion is qualitatively described and the normal form announced by Myhill is asserted. A calculus of lambda-conversion (Church) is described. Finally 10 pp. are devoted to the development of a new version of a (no quantifiers) recursive arithmetic due to the author and to Curry. From certain simple inductive "equalizing" rules, together with defining equations for $+$ and $\times$, many arithmetic results, including nullity of each representing function of a tautology, are derivable.

In Chapter IV (16 pp.) on *incompleteness of arithmetic*, Gödel's original incompleteness proof is traced through in outline and then his second incompleteness theorem on consistency is briefly discussed. Skolem's non-standard model for recursive arithmetic is presented. Unsolvability of decision-with-respect-to-provability for Z_f is obtained via an outline of the elegent Mostowski-Robinson-Tarski method. The key step of numeralwise expressibility of recursive functions is assumed. Unsolvability for Z and for predicate calculus are then deduced.

The final Chapter V (9 pp.) on *extended predicate calculus* leads from the Russell paradox through an outline description of Quine's system of *Mathematical Logic*, and concludes with a brief but good discussion of Cantor's theorem versus the Skolem-Löwenheim theorem (the Skolem "paradox").

Four pages of notes and bibliography are appended.

As quoted above, the book aims to fill, in a fairly orthodox way, the notorious summary-for-the-general-mathematician gap in present elementary logic texts. The selection of topics is on the whole excellent—as good as in any comparable volume. The reviewer furthermore approves in principle of the author's attempt to communicate by well-chosen example rather than exhaustive catalogue. Success in such an enterprise requires care and a sure expository touch. On various occasions the author exhibits these to an admirable degree. His summary discussions on the Heyting calculus, the Skolem "paradox," and in the Introduction are excellent. He is also impressively good at presenting summary proofs of certain combinatorial results—the expressibility of primitive recursive functions via the remainder theorem, properties of the Skolem non-standard model, and the lemmas on primitive recursive functions.

Unfortunately the book has faults that prevent the reviewer from giving it his full approval. Perhaps the most serious defect is a failure to exercise proper care in presenting the logistic method to a new reader. Unhappily this fault goes rather deeper than mere neglect of the use-mention distinction. Object-variables and meta-variables are introduced and then confused in an irregular way. Similarly (and in consequence) notions of axiom and axiom schema become confused. The author himself falls victim to this when he sets up what appears to be a predicate calculus with infinitely many axioms (there is no predicate substitution rule) and then on several later occasions (both Gödel theorem proofs) assumes finite axiomatization. Style of notation also varies from section to section without warning; Greek letters are number and function variables in IV, are meta-variables in presentation of both Quine systems. Semi-technical terms, e.g. "conjunction" and "classification index of signs" are occasionally introduced without definition.

The reviewer has the following further comments.

Chapter I. Statement of the deduction theorem is erroneous, since no restriction is made on substitution in hypotheses; validity and provability are confused in the alleged proof, so also (as on other occasions) are object- and meta-variables. In the natural inference formulation, the system is not complete as asserted (no negated formula is provable); furthermore rules for handling multiple antecedents and succedents are omitted: in the presence of a good illustration this latter is forgivable, but the omission should be acknowledged.

Chapter II. The substitution rule for sentence variables is incompatible with the definition of formula. The deduction theorem again omits a restriction on substitution for sentence variables (though one is made on individual variables). Simultaneous satisfiability is not defined for Henkin's proof, and in corollary G_6 validity and satisfiability are confused.

Chapter III. Axiom S_3 should either be omitted from Z, or the appropriate additional unicity axioms should be made explicit in Z_f; to speak of replacement of S_3 by S_3^* is misleading. Transcription from Church of rule C_1 for lambda-conversion is erroneous. The axiomatic basis for the recursive arithmetic is not made plain: if, as asserted, our theory has defining equations for *all* recursive functions, then it is no longer recursively enumerable; and it is not clear from the words "we notice the schema . . . " whether the induction schema is axiomatic or being asserted to be a derived rule. The treatment of recursive functions is open to two criticisms: (1) the class of general recursive functions and its role in foundations are not given sufficiently distinct expository emphasis, and (2) the ordinal normal form for recursive functions announced by Myhill is asserted (with a key typographical error—put "$f(\gamma(x))$" for "$\gamma(x)$") without direct reference or authority. The reviewer knows of no published proof of this result,—and believes that in such circumstances, even in an elementary text, appropriate authority (e.g. "correspondence") or qualification should be indicated. Indeed the reviewer would wish for more complete and specific references in general in a work where so much is asserted without proof.

Chapter IV. The discussion of the second incompleteness theorem is erroneous: two statements of the forms "'(for all x) [x not proof of f]' is provable" and "(for all x) ['x not proof of f' is provable]" are confused. The latter, asserted to be unprovable in Z, can be proved rather simply in Z by considering the alternative cases that Z be consistent or inconsistent. In introducing the proof of Tarski et al., "validity" is used, without definition, in their sense (=provability) rather than in any sense previously explained by the author.

Chapter V. The syntactical role of abstraction is not clear. Contextual definition should be earlier and more complete. Definition and use of "$y(x)$" on p. 95 is inconsistent; the definition is appropriate to a function but not to a general relation.

A final comment. The reviewer would have preferred that indication be given of logic-algebra and general recursive function theory as two of the most active research areas in foundations today, though the author might reasonably maintain that this is beyond his announced historical aims.

Among the less trivial typographical errors:

p. 45, "Sx" for "x" in last two occurrences in l. 16;
p. 67, l. 10b, "sentence calculus" for "predicate calculus";
p. 69, ll. 5–6, "$P=Q$" for "$G=0$" and "$F=G$" for "G";
p. 95, l. 13, "$\rightarrow$" for "&".

HARTLEY ROGERS, JR.

Geometric algebra. By E. Artin. New York, Interscience Publishers, Inc., 1957. 10+214 pp. $6.00.

When Hilbert's *Grundlagen der Geometrie* and other texts on the foundations of geometry appeared around the turn of the century, the approach was almost purely geometric. It is typical of the development of mathematics in the intervening years that, in this latest book on geometry, the approach is almost entirely algebraic.

In the preface the author states that his aim is to offer a text (based on lecture notes of a course he has given at New York University) which would be of a geometric nature yet distinct from a course in linear algebra, topology, differential geometry, or algebraic