CONTINUATION AND REFLECTION OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS[1]

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A solution of an ordinary differential equation can be continued as long as its graph stays in the domain, in which the equation is regular. On the other hand a solution of a partial differential equation can have a *natural boundary* interior to the domain of regularity of the equation. Let R be a closed region and S a portion of the boundary of R. Then S is a natural boundary for a solution u defined in R, if there exists no solution defined in a full neighbourhood of a point of S which agrees with u in R. Examples for the occurrence of such natural boundaries are well known from the theory of harmonic functions. Neither the equation nor the solution has to show any very singular behavior on approaching a natural boundary S.

Only in very special situations can one prove that *every* sufficiently regular solution known in a region R can be continued across a portion S of the boundary. This is e.g. the case for solutions of a single differential equation which is hyperbolic with respect to S. It is also the case for solutions of certain *overdetermined* systems of equations, like those associated with analytic functions of several complex variables; other systems with this property have been studied by S. Bochner [1].

In more general cases the only solutions for which one can prove continuibility are those satisfying *suitable boundary* conditions on S. The classical example is furnished by solutions $u(x, y)$ of the Laplace equation

$$u_{xx} + u_{yy} = 0 \tag{1a}$$

defined for $y \geqq 0$ and satisfying the boundary condition

$$u(x, 0) = 0. \tag{1b}$$

They can always be continued across $y=0$ by the formula

$$u(x, -y) = -u(x, y). \tag{1c}$$

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[1] A summary of the results of this paper appeared in *Colloques internationaux du centre national de la recherche scientifique*, vol. 71, Nancy, 1956.

In recent years one has succeeded in extending this continuation theorem in various ways, involving either more general boundary conditions or more general differential equations. I mention the following generalizations. Thus a sufficiently regular solution of (1a) defined for $y \geqq 0$ can be continued across $y=0$ whenever it satisfies instead of (1a) a boundary condition of the form

$$u_y = A(x, u, v, u_x) \tag{1d}$$

on $y=0$, where A is an analytic function of its arguments and v denotes the conjugate harmonic to u (See H. Lewy [2], R. Gerber [3]). A solution of any linear elliptic equation of any order and in any number of dimensions which has vanishing Dirichlet data on a hyperplane can be continued across that plane, provided the equation has analytic coefficients. (See Morrey [5], Nirenberg and Morrey [4].) A solution of the equation of minimal surfaces that is analytic on an analytic boundary curve can be continued across that curve. (See Lewy [15].)

Continuation of solutions constitutes an important tool in finding an explicit solution of many problems of applied mathematics. It has been used by Helmholtz, Kirchhoff, Shiffman and others in determining motions of liquids possessing a free boundary.[2] In these problems the continuation of the flow is *not explicit* as long as the location of the free boundary is not known; the free boundary and the flow are obtained simultaneously by also taking into account the boundary conditions on fixed boundaries and reducing the problem to one in conformal mapping. Explicit local continuation can be used, when the boundary conditions are sufficiently simple. This situation arises e.g. in the linearized theory of gravity waves, where one is led to consider solutions of the Laplace equation (1a), which satisfy a boundary condition of the form

$$u_y + ku = 0 \tag{2}$$

with constant k on the x-axis.[3] The continuation of solutions of more general second order equations with constant coefficients which satisfy boundary conditions of the type (2) on a plane has been studied by Diaz and Ludford [9].

A few cases in which solutions of higher order equations can be continued by an explicit reflection formula have been established. Poritsky [10] proved that a solution $u(x, y)$ of the bi-harmonic equation

$$\Delta^2 u = 0 \tag{3a}$$

[2] See Lamb [6], Ch. IV, Milne-Thomson [7].

[3] See Stoker [8].

which satisfies the boundary conditions

$$u = u_y = 0 \qquad \text{for } y = 0 \tag{3b}$$

can be continued across the x-axis using the formula

$$u(x, -y) = - u(x, y) + 2yu_y(x, y) - y^2\Delta u(x, y). \tag{3c}$$

Analogous formulae have been obtained by Duffin [**11**; **12**] for the systems of equations describing static equilibrium of a 3-dimensional isotropic elastic body. Finally Huber [**13**] has generalized formula (3c) to the case of solutions of a poly-harmonic equation

$$\Delta^p u(x_1 \cdots, x_n) = 0 \tag{4}$$

having vanishing Dirichlet data on a hyper-plane.

It seems desirable to have a more systematic study of the types of boundary conditions that guarantee existence of a continuation of the solution of a differential equation and also of the cases in which such continuation can be achieved by an explicit formula. Of course, such formulae can be expected only for very simple situations, say for linear equations with constant coefficients and for solutions satisfying linear homogeneous boundary conditions with constant coefficients on a plane boundary. I should like to present here some results in this direction for a rather simplified situation: The equation shall be linear and homogeneous and have constant coefficients, the boundary shall be plane, and the boundary conditions shall be of the type that the solution and a certain number of its derivatives vanish on the boundary plane. The results have been extended by R. I. Canavan to the most general equation with constant coefficients in his thesis on *Necessary conditions for continuation and reflection principles for solutions of linear partial differential equations with constant coefficients*, New York University, 1957.

There shall be $n+1$ independent variables $x_0, x_1, \cdots, x_n$. We write x for the n-vector $(x_1, \cdots, x_n)$, and $u(x_0, x)$ for the function $u(x_0, x_1, \cdots, x_n)$. The differential equation shall be of the form

$$P(\xi_0, \xi)u(x_0, x) = 0 \tag{5a}$$

where $P(\xi_0, \xi)$ is a form with constant complex coefficients of degree N in

$$\xi_0 = \partial/\partial x_0 \quad \text{and} \quad \xi = (\partial/\partial x_1 \cdots, \partial/\partial x_n).$$

Moreover we make the assumption

$$P(1, 0) = 1, \tag{5b}$$

equivalent to noncharacteristic character of the plane $x_0=0$. The boundary condition shall be of the form

$$(\overset{k}{\xi_0}u)_{x_0=0} = 0 \qquad \text{for } k = 0, \cdots, s-1. \tag{5c}$$

We shall show that in general if s is too small there exist solutions of (5a, c) which are in C_∞ for $x_0 \geqq 0$ and have the plane $x_0=0$ as natural boundary.

Solutions of partial differential equations with natural boundaries can be obtained in the form of series similar to those used for constructing analytic functions with natural boundaries. We consider the characteristic equation of equation (5a):

$$P(\eta_0, \eta) = 0. \tag{6}$$

For given $\eta=(\eta_1, \cdots, \eta_n)$ the N roots η_0 of (6) may be denoted by $\lambda_1, \cdots, \lambda_N$. Assume now that for a certain real η with $\eta\cdot\eta=1$ the roots η_0 of (6) satisfy the conditions

$$\operatorname{Im} \lambda_1 \geqq \operatorname{Im} \lambda_2, \cdots, \operatorname{Im} \lambda_{s+1} \geqq 0 \tag{7a}$$

and

$$\operatorname{Im} \lambda_1 > 0. \tag{7b}$$

Let $f(x_0)$ be the solution of the ordinary differential equation

$$\prod_{k=1}^{s+1} (\xi_0 - i\lambda_k) f(x_0) = 0$$

with initial conditions

$$\overset{k}{\xi_0} f(x_0) = \begin{cases} 0 & \text{for } k = 0, \cdots, s-1, \\ 1 & \text{for } k = s \end{cases}$$

for $x_0=0$. Then, because of (7a), $f(x_0)$ and any of its derivatives become infinite at most like a power of x_0 for $x_0 \to +\infty$. The function

$$e^{i\eta\cdot x} f(x_0)$$

is a solution of (5a, c). It follows that

$$u(x_0, x) = \sum_{j=0}^{\infty} e^{-j^2} \exp(i2^j\eta\cdot x) f(2^j x_0) \tag{8}$$

for $x_0 \geqq 0$ is again a solution of (5a, c) and is of class C_∞. We shall prove that this function u has $x_0=0$ as natural boundary.

Since $\eta \neq 0$ there exists a real vector y such that

$$y \cdot \eta = 2\pi.$$

We have for any positive integer r and for $x_0 \geqq 0$ from (8)

$$u(x_0, x + 2^{-r}y) - u(x_0, x) = w_r(x_0, x), \tag{9}$$

where

$$w_r(x_0, x) = \sum_{j=0}^{r-1} e^{-j^2} \exp(i2^j\eta \cdot x) f(2^j x_0)(\exp(i2^{j-r+1}\pi) - 1) \tag{9a}$$

is an entire analytic solution of (5a).

If $u(x_0, x)$ would not have the plane $x_0 = 0$ as natural boundary there would exist a point $x = X$, a positive number ϵ, and a function $U(x_0, x)$ of class C_N in

$$|x - X| < \epsilon, \qquad |x_0| < \epsilon \tag{10}$$

which satisfies (5a) and agrees with $u(x_0, x)$ for $x_0 \geqq 0$.

The expression

$$V(x_0, x) = U(x_0, x + 2^{-r}y) - U(x_0, x) - w_r(x_0, x)$$

is a solution of (5a) defined for

$$|x - X| < \epsilon/2, \qquad |x_0| < \epsilon$$

provided r is so large that

$$2^r > 2|y|/\epsilon.$$

Moreover $V(x_0, x) \equiv 0$ for $x_0 \geqq 0$. It follows from the uniqueness theorem of Holmgren[4] that there exists a positive ϵ', independent of r, such that $V(x_0, x) = 0$ for all x_0, x with

$$|x - X| < \epsilon', \qquad |x_0| < \epsilon'.$$

In other words, the functional equation (9) satisfied by u must be preserved under continuation.

Let $Q(\xi_0, \xi)$ denote the differential operator

$$Q(\xi_0, \xi) = \prod_{k=2}^{s+1} (\xi_0 - \lambda_k \eta \cdot \xi).$$

Since

$$\prod_{k=2}^{s+1} (\xi_0 - i\lambda_k) f(x_0) = e^{i\lambda_1 x_0},$$

[4] See John [14].

we have for $|x-X|<\epsilon/2,\ -\epsilon<x_0<0$

$$
\begin{aligned}
&| Q(\xi_0, \xi)[U(x_0, x+2^{-r}y) - U(x_0, x)] | \\
&= \left| \sum_{j=0}^{r-1} 2^{sj}e^{-j^2} \exp(i2^j x\cdot\eta)(\exp(i2^{j-r+1}\pi) - 1) \exp(i2^j\lambda_1 x_0) \right| \\
&\geqq 2\cdot 2^{s(r-1)} \exp(-(r-1)^2 - 2^{r-1}x_0 \operatorname{Im}(\lambda_1)) \\
&\quad - 2\exp(-2^{r-2}x_0 \operatorname{Im}(\lambda_1)) \sum_{j=0}^{\infty} 2^{sj}e^{-j^2}.
\end{aligned}
$$

Hence

$$
\lim_{r\to\infty} | Q(\xi_0, \xi)[U(x_0, x+2^{-r}y) - U(x_0, x)] | = \infty .
$$

But this is incompatible with $U(x_0, x)$ belonging to class C_N. Consequently $u(x_0, x)$ has the plane $x_0=0$ as natural boundary, if we can find a real η with $|\eta|=1$ such that the corresponding roots λ_k of the characteristic equation (6) satisfy (7a, b). Since equation (6) is homogeneous in η_0 and η, the restriction $|\eta|=1$ is unessential.

We call boundary conditions for a solution on the plane $x_0=0$ *adequate* for continuation, if every sufficiently regular solution defined on one side of the plane in a neighborhood of the plane can be continued locally across that plane. We can then formulate our result as follows:

For the boundary conditions (5c) *to be adequate for continuation of a solution of* (5a) *it is necessary that for every real* η *the roots* λ_k *are either all real, or that at least* $N-s$ *of them have positive real part and at least* $N-s$ *of them have negative real part.*

Consequently for $s<N/2$, the boundary conditions (5c) can be adequate for continuation only if for every real η the roots λ_k are all real, or, in other words, they are never adequate unless the differential equation (5a) is hyperbolic with respect to the plane $x_0=0$.

If N is even and $s=N/2$, conditions (5c) express that u has vanishing Dirichlet data on $x_0=0$. Then the vanishing of the Dirichlet data can be adequate for continuation only if for any real η the solutions η_0 of (6) are either all real or all imaginary. They are all imaginary when equation (5a) is *elliptic.* In this case vanishing of the Dirichlet data is adequate for continuation by virtue of the result of Morrey and Nirenberg mentioned above. It is possible that the necessary condition for adequacy of boundary data is always sufficient, regardless of the type of the equation.

We apply the condition to the example of the equation of elastic waves in an istropic medium:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\left(\frac{\partial^2}{\partial t^2} - c'^2\Delta\right)u(x, y, z, t) = 0 \tag{11a}$$

where Δ is the Laplace operator in xyz-space. Here the boundary conditions

$$u = \frac{\partial u}{\partial x} = 0 \qquad \text{for } x = 0 \tag{11b}$$

are inadequate for continuation, if the constants c and c' are distinct. For, in that case, there exist real y, z, t such that the 4 roots of the characteristic equation

$$(t^2 - c^2(x^2 + y^2 + z^2))(t^2 - c'^2(x^2 + y^2 + z^2)) = 0$$

are neither all real nor all imaginary. If, on the other hand, $c=c'$, the conditions (11b) are adequate; indeed, then, continuation is achieved by the explicit formula (analogous to (3c))

$$u(-x, y, z, t) = -u + 2xu_x + \frac{x^2}{c^2}\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)u \tag{11c}$$

where on the right hand side the arguments of u are x, y, z, t.

Equation (11a) is hyperbolic with respect to the plane $t=0$. Thus the boundary conditions

$$u = u_t = 0 \qquad \text{for } t = 0 \tag{11d}$$

are adequate for continuation, even when $c \neq c'$. Here again an explicit "reflection formula" exists, giving the continuation, which will be derived later (see formula (32c)).

We next turn to the question: Can continuation be achieved by a simple formula of the type (1c) or (3c), if the data are adequate for continuation? Instead of trying to give directly a more precise meaning to what constitutes a "simple formula" it is convenient to take notice of a more qualitative feature of formulae (1c), (3c), namely that they constitute "reflection principles." The values of the continued function at one point P are expressible in terms of the values of the original functions and of its derivatives in points on the line through P perpendicular to the boundary plane. We shall restrict ourselves to the case, where the order N of the differential equation is even, $N=2m$, and where the boundary conditions on the plane $x_0=0$ consist in the vanishing of the Dirichlet data ($s=m$):

$$(\overset{k}{\xi_0}u)_{x_0=0} = 0 \qquad \text{for } k = 0, \cdots, m-1. \tag{12}$$

We are led to the following definition: Equation (5a) is said to possess a reflection principle (with respect to the Dirichlet data on the plane $x_0=0$), if there exists a number $\epsilon>0$ (depending only on P) such that every solution $u(x_0, x)$ of (5a), which is analytic in a cylindrical set which has the domain D in x-space as base

$$x \text{ in } D, \qquad 0 \leqq x_0 < h \tag{13a}$$

can be continued as a regular analytic solution into the set

$$x \text{ in } D, \; -\epsilon h < x_0 \leqq 0, \tag{13b}$$

provided u satisfies the boundary conditions (12) for x in D.

It turns out that all equations having a reflection principle in this sense can be characterized rather simply, and that existence of a *reflection principle* implies existence of an *explicit formula* for continuation of a solution.

It follows from the definition that an equation (5a) possesses no reflection principle, if there exists an analytic solution $u(x_0, x)$ of (5a) with vanishing Dirichlet data on $x_0=0$, which is regular for $x=0$, $x_0\geqq 0$, and has a singularity in some point of the negative x_0-axis. Let now η be any *complex* n-vector$\neq 0$, and let $\lambda_1, \cdots, \lambda_{2m}$ be the corresponding roots η_0 of (6). Then

$$\begin{aligned} u(x_0, x) &= \frac{x_0^m}{\prod_{k=1}^{m+1} (\eta\cdot x + \lambda_k x_0 + \lambda_1)} \\ &= \frac{(-1)^{m+1}}{2\pi i} \oint_C \frac{d\lambda}{(\eta\cdot x + \lambda x_0 + \lambda_1)\prod_{k=1}^{m+1} (\lambda - \lambda_k)}, \end{aligned}$$

(where C is a small path about the point $\lambda = -(\eta\cdot x+\lambda_1)/x_0$) is easily seen to be an analytic solution of (5a), (12) which is singular at the point

$$x = 0, \qquad x_0 = -1.$$

If this solution is regular for $x=0$, and $x_0\geqq 0$, there is no reflection principle. This is the case, when

$$\lambda_k x_0 + \lambda_1 \neq 0 \quad \text{for real } x_0 \geqq 0 \text{ and } k = 1, \cdots, m+1.$$

Thus necessary for the existence of a reflection principle is that whenever one root λ_1 is different from 0 then at least one out of any m of the remaining $2m-1$ roots lies on the ray opposite to that containing λ_1. In other words, it is necessary that for any complex η either the roots λ_k of (6) all vanish, or that none of them vanishes and that they all lie on the same line through the origin, half of them on either side of the origin.

We are led to look for the forms $P(\eta_0, \eta)$ whose roots $\eta_0=\lambda_k$ lie on a straight line through the origin for any complex η. There would have to exist a real θ and real c_k such that

$$\lambda_k = c_k e^{i\theta} \qquad \text{for } k = 1, \cdots, 2m.$$

Let

$$P(\eta_0, \eta) = \eta_0^{2m} + q_1(\eta)\eta_0^{2m-1} + q_2(\eta)\eta_0^{2m-2} + \cdots + q_{2m}(\eta),$$

where $q_k(\eta)$ is a form of degree k in $\eta=(\eta_1, \cdots, \eta_n)$; $q_k(\eta)e^{-ik\theta}$ would have to be real for any $k=1, \cdots, 2m$. Hence for any complex η and any $k, j=1, \cdots, 2m$

$$\frac{(q_k(\eta))^j}{(q_j(\eta))^k} \tag{14}$$

would have to be real, whenever $q_j(\eta)\neq 0$. It follows that the expression (14) must be constant, since an analytic function of η which is real for all complex η in the neighbourhood of a point in η-space must be a constant.

Let S be the subset of the set of integers $j=1, \cdots, 2m$, for which $q_j(\eta)\not\equiv 0$. Unless $P\equiv\eta_0^{2m}$, the set S is not empty. Let μ be the greatest common divisor of all the numbers in S. We can represent μ in the form

$$\mu = \sum_{j\subset S} jn_j$$

with integral coefficients n_j. Let $Q=Q(\eta)$ be the rational function defined by

$$Q = \prod_{j\subset S} q_j(\eta)^{n_j}.$$

Then for any k in S

$$\frac{q_k(\eta)^\mu}{Q(\eta)^k} = \prod_{j\subset S} \frac{q_k(\eta)^{jn_j}}{q_j(\eta)^{kn_j}} = \text{constant} = (\gamma_k)^\mu$$

or

$$q_k(\eta) = \gamma_k(\eta)Q(\eta)^{k/\mu} \tag{15}$$

where $(\gamma_k)^\mu$ is independent of η and $\neq 0$. Since the polynomial $q_k(\eta)$

is bounded for all bounded η, the same holds for the rational function $Q(\eta)$, which consequently must be a polynomial in η. The number k/μ is an integer for k in S, and thus, by (15), γ_k is independent of η. It follows that P is of the form

$$P = \sum_{0 \leqq r \leqq 2m/\mu} \eta_0^{2m-r\mu} c_r Q(\eta)^r. \tag{16}$$

Here the c_r are constant, and Q is a form of degree μ. Whenever $\eta_0 = \lambda$ is a root of the polynomial P given by (16), $\epsilon\lambda$ is also a root, where ϵ is any μth root of unity. For $\lambda \neq 0$, $\mu > 2$, the various $\epsilon\lambda$ do not all lie on a straight line. Hence P is either given by η_0^{2m} or by (16), where Q is a form of degree μ with $\mu = 1$ or $\mu = 2$. In either case we can factor P, and conclude that a reflection principle is possible only if P is of one of the following 3 types:

$$P = \eta_0^{2m}, \tag{17a}$$

$$P = \prod_{k=1}^{2m} (\eta_0 - a_k L(\eta)), \tag{17b}$$

$$P = \prod_{k=1}^{m} (\eta_0^2 - a_k^2 q(\eta)). \tag{17c}$$

Here the a_k are constant, and L and q denote, respectively, a linear or a quadratic form in η which is independent of k. Since the roots η_0 must either vanish or lie on one line through the origin, half on each side, we can change L or q by a constant factor, so that in (17b) the a_k are all real, $a_1, \cdots, a_m$ positive, $a_{m+1}, \cdots, a_{2m}$ negative, and in (17c) so that all the a_k are real and positive.

It remains to show that for any equation of one of the types (17a, b, c) with real a_k of the correct sign there actually exists a reflection principle. This is achieved by constructing an explicit formula that represents a continuation of the desired type. We omit the case of an equation of type (17a) in which continuation is trivial from further consideration. To arrive at reflection formulae for equations of type (17b, c), we first derive such a formula for the case of an ordinary differential equation with constant coefficients.

Let $p(\lambda)$ be a polynomial of degree $2m$ with roots $a_1, \cdots, a_{2m}$, where $a_k \neq a_j$ for $k = 1, \cdots, m$ and $j = m+1, \cdots, 2m$, and all the a_i are different from 0. We consider a solution $u(x_0)$ of the equation

$$P(\xi_0)u(x_0) = 0 \tag{18a}$$

for which

$$(\xi_0^k u(x_0))_{x_0=0} = 0 \qquad \text{for } k = 0, 1, \cdots, m-1. \tag{18b}$$

We put

$$A(\lambda) = \prod_{k=1}^{m} (\lambda - a_k), \qquad B(\lambda) = \sum_{k=m+1}^{2m} (\lambda - a_k), \tag{18c}$$

so that $p(\lambda) = A(\lambda)B(\lambda)$. Let C' and C'' be two closed paths in the λ-plane not containing the origin which are exterior to each other, and such that C' contains $a_1, \cdots, a_m$ and C'' contains $a_{m+1}, \cdots, a_{2m}$. We define for any λ, μ the polynomial $R(\lambda, \mu)$ by

$$R(\lambda, \mu) = \frac{p(\lambda) - p(\mu)}{\lambda - \mu}.$$

Then for any integer $j \geqq 0$ the identity

$$\begin{aligned} \xi_0^j u(x_0) = {} & (2\pi i)^{-2} \oint_{C'} d\alpha \oint_{C''} d\beta \\ & \cdot \frac{\beta^j R\left(\frac{\alpha}{\beta}\xi_0, \alpha\right) u\left(\frac{\beta}{\alpha} x_0\right) - \alpha^j R\left(\frac{\beta}{\alpha}\xi_0, \beta\right) u\left(\frac{\alpha}{\beta} x_0\right)}{(\beta - \alpha) A(\alpha) B(\beta)} \end{aligned} \tag{19}$$

is satisfied for every solution u of (18a, b). The expression on the righthand side of (19) is clearly defined since any solution u of (18a) is analytic, and $(R(\alpha/\beta)\xi_0, \alpha)$ is a differential operator depending regularly on the parameters α, β.

Identity (19) can be verified most easily by observing that the general solution u of (18a, b) is of the form

$$u(x_0) = \frac{1}{2\pi i} \oint_C \frac{h(\lambda) e^{\lambda x_0}}{p(\lambda)} d\lambda, \tag{20}$$

where $h(\lambda)$ is an arbitrary polynomial in λ of degree $< m$, and C is a path containing all roots of p. We may assume that C contains both C' and C'' in its interior. Then

$$\begin{aligned} R\left(\frac{\alpha}{\beta}\xi_0, \alpha\right) u\left(\frac{\beta}{\alpha} x_0\right) &= \frac{1}{2\pi i} \oint_C \frac{p(\lambda) - p(\alpha)}{(\lambda - \alpha) p(\lambda)} h(\lambda) e^{\beta\lambda x_0/\alpha} d\lambda \\ &= h(\alpha) e^{\beta x_0} - \frac{p(\alpha)}{2\pi i} \oint_C \frac{h(\lambda)}{(\lambda - \alpha) p(\lambda)} e^{\beta\lambda x_0/\alpha} d\lambda. \end{aligned}$$

The second term makes no contribution, since

$$\frac{p(\alpha)}{(\beta - \alpha) A(\alpha)(\lambda - \alpha)} e^{\beta\lambda x_0/\alpha}$$

is a regular function of α for α inside C', β on C'', λ on C and the point $\alpha=0$ outside C'. Moreover, from the theory of residues

$$\frac{1}{2\pi i}\oint_{C'} \frac{h(\alpha)}{(\beta-\alpha)A(\alpha)}\,d\alpha = \frac{h(\beta)}{A(\beta)},$$

since h is of lower degree than A, and $\alpha=\beta$ is the only pole of the integrand outside C'. It follows that

$$(2\pi i)^{-2}\oint_{C'} d\alpha \oint_{C''} d\beta\, \frac{\beta^j R\left(\frac{\alpha}{\beta}\xi_0,\alpha\right)u\left(\frac{\beta}{\alpha}x_0\right)}{(\beta-\alpha)A(\alpha)B(\beta)} = \frac{1}{2\pi i}\oint_{C''}\frac{\beta^j e^{\beta x_0}h(\beta)}{p(\beta)}d\beta.$$

Evaluating the remaining portion of the right hand side of (19), in a similar way, the validity of the identity (19) becomes obvious if we compare it with the expression for $\xi_0^j u(x_0)$ arising from (20).

In principle the right hand side of identity (19) can be evaluated by the theory of residues. It then yields a representation of $\xi_0^j u(x_0)$ in terms of u and its derivatives taken for the arguments $a_k x_0/a_j$ and $a_j x_0/a_k$ where $k=1,\cdots,m$ and $j=m+1,\cdots,2m$. If all a_k are positive all a_j negative, this represents a reflection principle, since all a_k/a_j are then negative and since for $x_0<0$ we can express $\xi_0^j u(x_0)$ in terms of u and its derivatives for positive arguments. If all a_k, a_j are distinct we have, for example, the formula

$$\xi_0^j u(x_0) = \sum_{k=1}^{m}\sum_{r=m+1}^{2m} \tag{20a}$$
$$\frac{a_r^j R\left(\frac{a_k\xi_0}{a_r},\ a_k\right)u\left(\frac{a_r x_0}{a_k}\right) - a_k^j R\left(\frac{a_r\xi_0}{a_k},\ a_r\right)u\left(\frac{a_k}{a_r}x_0\right)}{(a_r-a_k)A'(a_k)B'(a_r)}.$$

Here

$$R\left(\frac{a_k\xi_0}{a_r},\ a_k\right)u\left(\frac{a_r x_0}{a_k}\right) = \left[\frac{p(d/ds)}{d/ds - a_k}u(s)\right]_{s=a_r x_0/a_k}.$$

If we differentiate identity (19) k times with respect to x_0 and put $x_0=0$ we obtain the relation

$$[\xi_0^{j+k}u(x_0)]_{x_0=0} = \left[F_{jk}\left(\frac{d}{ds}\right)u(s)\right]_{s=0} \tag{21}$$

where $F_{jk}(\lambda)$ is a polynomial of degree $\leqq 2m-1+k$:

$$(22)\quad F_{jk}(\lambda) = (2\pi i)^{-2}\oint_{C'} d\alpha \oint_{C''} d\beta \frac{\lambda^k[\beta^{k+j}\alpha^{-k}R(\lambda, \alpha) - \alpha^{k+j}\beta^{-k}R(\lambda, \beta)]}{(\beta - \alpha)A(\alpha)B(\beta)}.$$

There exist polynomials $G_{jk}(\lambda)$ and $H_{jk}(\lambda)$ such that

$$(23)\quad \lambda^{j+k} = F_{jk}(\lambda) + G_{jk}(\lambda) + H_{jk}(\lambda)p(\lambda)$$

where G_{jk} is of degree $<2m$. Then from (21) we have

$$[G_{jk}(\xi_0)u(x_0)]_{x_0=0} = 0$$

for any solution u of (18a, b). Since for such a u the values of $[\xi_0^k u(x_0)]_{x_0=0}$ vanish for $k=0, \cdots, m-1$ and are arbitrary for $k=m, m+1, \cdots, 2m-1$, it follows that G_{jk} must be of degree $<m$. Hence there exists an identity of the form (23) with a polynomial G_{jk} of degree $<m$.

We are now in a position to write down a reflection principle for an equation of the type (17b). Let $u(x_0, x)$ be a solution of the differential equation

$$(24)\quad P(\xi_0, \xi)u(x_0, x) = \prod_{k=1}^{2m} (\xi_0 - a_k L(\xi))u(x_0, x) = 0$$

which is analytic for real x_0, x in a set (13a), and satisfies the Dirichlet conditions (12) for x in D. Here the a_k shall be real numbers, $a_1, \cdots, a_m$ positive, $a_{m+1}, \cdots, a_{2m}$ negative, and L shall be a linear form with constant (real or complex) coefficients.

Formally the conditions on u are of the type (18a, b), only with a_k replaced by $a_k L(\xi)$. For $j=2m-1$, we are then led to the identity

$$(25)\quad \xi_0^{2m-1} u(x_0, x) = v(x_0, x)$$

where

$$(26\text{a})\quad v(x_0, x) = (2\pi i)^{-2}\oint_{C'} d\alpha \oint_{C''} d\beta \frac{S}{(\beta - \alpha)A(\alpha)B(\beta)},$$

$$(26\text{b})\quad S = \beta^{2m-1} \frac{P\left(\frac{\alpha}{\beta}\xi_0, \xi\right) - P(\alpha L(\xi), \xi)}{\frac{\alpha}{\beta}\xi_0 - \alpha L(\xi)} u\left(\frac{\beta}{\alpha}x_0, x\right) - \alpha^{2m-1} \frac{P\left(\frac{\beta}{\alpha}\xi_0, \xi\right) - P(\beta L(\xi), \xi)}{\frac{\beta}{\alpha}\xi_0 - \beta L(\xi)} u\left(\frac{\alpha}{\beta}x_0, x\right).$$

Here the polynomials $A(\alpha)$, $B(\beta)$ are defined by (18c); the path C' can be chosen close to the smallest interval on the positive real axis containing the points $a_1, \cdots, a_m$ and C'' close to the smallest interval on the negative real axis containing the points $a_{m+1}, \cdots, a_{2m}$. Then α/β and β/α are always close to a negative real number whose absolute value does not exceed the largest of the values $-a_k/a_j$. Let ϵ be the positive number defined by

$$\frac{1}{\epsilon} = \operatorname*{Max}_{k,j=1,\cdots,2m} \frac{-a_k}{a_j}.$$

Then for α on C', β in C'' the quantities α/β and β/α will be close to negative real numbers lying in the interval from $-1/\epsilon$ to 0. Furthermore, for x in D and $-\epsilon h < x_0 \leqq 0$ the quantities $u((\alpha/\beta)x_0, x)$ and $u((\beta/\alpha)x_0, x)$ will be defined as well as S which represents a differential expression on u and depends regularly on the parameters α, β. (The corresponding formula for $j < 2m-1$ would still contain the factor $L(\xi)^{j-2m+1}$, which would not be a *differential* operator.) It follows that the function $v(x_0, x)$ is defined and analytic for real x_0, x in the set (13b). In order to prove that it represents a continuation of $\xi_0^{2m-1}u$ it is sufficient to compare the derivatives with respect to x_0 at $x_0 = 0$. Comparison with formula (22) shows that this amounts to proving the identity

$$(27) \quad [\xi_0^{2m-1+k} u(x_0, x)]_{x_0=0} = \left[L^{2m-1+k}(\xi) F_{2m-1\,k}\left(\frac{1}{L(\xi)} \frac{d}{ds}\right) u(s, x)\right]_{s=0}$$

for $j = 2m-1$, $k = 0, 1, 2, \cdots$ and any solution u of (24), (12). By (23) we have

$$\xi_0^{2m-1+k} = L^{2m-1+k} F_{2m-1\,k}(L^{-1}\xi_0) + L^{2m-1+k} G_{2m-1\,k}(L^{-1}\xi_0)$$
$$+ L^{k-1} H_{2m-1\,k}(L^{-1}\xi_0) L^{2m} p(L^{-1}\xi_0)$$

identically in ξ_0 and L. For $L = L(\xi)$ both sides represent differential operators. We apply this identity to a solution of (24), (12) obtaining (27) since

$$L^{2m}(\xi) p(L^{-1}(\xi)\xi_0) u(x_0, x) = P(\xi_0, \xi) u(x_0, x) = 0$$

and since also

$$[L^{m-1}(\xi) G_{2m-1\,k}(L^{-1}(\xi)\xi_0) u(x_0, x)]_{x_0=0} = 0,$$

$L^{m-1}(\xi) G_{2m-1\,k}(L^{-1}(\xi)\xi_0)$ being a polynomial of degree $< m$. This completes the proof of the fact that $v(x_0, x)$ represents an analytic continuation of $\xi_0^{2m-1}u(x_0, x)$.

The function v is given by a double integral in the complex α- and β-planes; the integrand has poles at the zeros of $A(\alpha)$ and $B(\beta)$. The integral can be expressed by the residues at these poles, which in turn can be written as differential operators acting on the integrand. In principle then $v(x_0, x)$ can be expressed as a combination of derivatives of u taken at the points

$$\left(\frac{a_k}{a_j} x_0, x\right) \quad \text{and} \quad \left(\frac{a_j}{a_k} x_0, x\right)$$

with $k=1, \cdots, m$ and $j=m+1, \cdots, 2m$. The resulting formula involves only u and a finite number of its derivatives in a number of real points, all lying on the same perpendicular to the plane $x_0=0$. In case all the a_i are distinct the formula is of the form (20a):

$$\xi_0^{2m-1} u(x_0, x) = \sum_{k=1}^{m} \sum_{r=m+1}^{2m} \frac{S_{kr}}{(a_r - a_k)A'(a_k)B'(a_r)},$$

$$S_{kr} = a_r^{2m-1}\left[\frac{P\left(\frac{d}{ds}, \xi\right) - P(a_k L(\xi), \xi)}{\frac{d}{ds} - a_k L(\xi)} u(s, \xi)\right]_{s=a_r x_0/a_k}$$

$$- a_k^{2m-1}\left[\frac{P\left(\frac{d}{ds}, \xi\right) - P(a_r L(\xi), \xi)}{\frac{d}{ds} - a_r L(\xi)} u(s, \xi)\right]_{s=a_k x_0/a_r}.$$

The identity (25) gives a continuation v of $\xi_0^{2m-1}u$. If we define a function $U(x_0, x)$ in (13b) by the conditions

$$(28) \qquad \xi_0^{2m-1} U = v, \qquad (\xi_0^k U)_{x_0=0} = (\xi_0^k u)_{x_0=0} \qquad \text{for } k = 0, \cdots, 2m-2,$$

U and u will agree with all their derivatives at $x_0=0$, and hence U will represent an analytic continuation of u. The determination of U in terms of v involves only repeated integration along the line $x_0=$ constant. This establishes the existence of a reflection principle in the sense defined here for equations of the type (17b). It is easily seen that the resulting formulae yield a continuation of a solution u of (24), (12) not only when u is analytic, but whenever u is sufficiently often differentiable.

Reflection principles for equations of type (17c) are obtained from the same basic formula (19), taken for $j=2m-2$. If $p(\lambda)$ is an *even* polynomial, with roots

$$\lambda = \pm a_k \qquad \text{for } k = 1, \cdots, m,$$

we have $B(\lambda) = (-1)^m A(-\lambda)$. Replacing β by $-\beta$ in (19) we can choose the same path of integration on C' for α and β. We then arrive at the formula

$$\xi_0^{2m-2} u(x_0) = \frac{2(-1)^m}{(2\pi i)^2} \oint_{C'} d\alpha \oint_{C'} d\beta \frac{\alpha \beta^{2m-2}}{(\alpha + \beta) A(\alpha) A(\beta)} \cdot \frac{p\left(\frac{\alpha}{\beta} \xi_0\right) - p(\alpha)}{\frac{\alpha^2}{\beta^2} \xi_0^2 - \alpha^2} u\left(\frac{-\beta}{\alpha} x_0\right). \tag{29}$$

Here for positive real $a_1, \cdots, a_m$ the path C' can be taken as a set of small circles about the points $a_1, \cdots, a_m$.

Let now $u(x_0, x)$ be a function which is analytic in the set (13a), satisfies the differential equation

$$P(\xi_0, \xi) u = \prod_{k=1}^{m} (\xi_0^2 - a_k^2 q(\xi)) u = 0 \tag{30}$$

and the Dirichlet conditions (12), and let

$$\begin{aligned} v(x_0, x) &= \frac{2(-1)^m}{(2\pi i)^2} \oint_{C'} d\alpha \oint_{C'} d\beta \frac{\alpha \beta^{2m-2} S}{(\alpha + \beta) A(\alpha) A(\beta)}; \\ S &= \frac{P\left(\frac{\alpha}{\beta} \xi_0, \xi\right) - P(\alpha (q(\xi))^{1/2}, \xi)}{\frac{\alpha^2}{\beta^2} \xi_0^2 - \alpha^2 q(\xi)} u\left(\frac{-\beta}{\alpha} x_0, x\right), \end{aligned} \tag{31}$$

then v is defined and analytic in the set (13b), where

$$\epsilon = \min_{k,j=1,\cdots,m} \frac{a_j}{a_k}.$$

By the same argument as before, comparing derivatives at $x_0 = 0$, one verifies that

$$\xi_0^{2m-2} u(x_0, x) = v(x_0, x).$$

Repeated integration with respect to x_0 then furnishes an explicit reflection formula for $u(x_0, x)$ itself.

A typical example is given by the case $m = 2$ with distinct a_k. Let $u(x_0, x)$ be a solution of the equation

$$(\xi_0^2 - a_1^2 q(\xi))(\xi_0^2 - a_2^2 q(\xi))u(x_0, x) = 0, \tag{32a}$$

where q is a quadratic form and a_1, a_2 are distinct real positive quantities. Let u satisfy the initial conditions

$$u = \xi_0 u = 0 \quad \text{for } x_0 = 0. \tag{32b}$$

The residues at the simple poles of the integrand in the expression for v are easily evaluated, and one finds the formula

$$\begin{aligned}(a_1 + a_2)(a_1 - a_2)^2 \xi_0^2 u(x_0, x) &= \left[(a_1 + a_2)\left((a_1^2 + a_2^2)\frac{\partial^2}{\partial s^2} - 2a_1^2 a_2^2 q(\xi)\right)u(s, x)\right]_{s=-x_0} \\ &\quad - 2a_1 a_2^2\left[\left(\frac{\partial^2}{\partial s^2} - a_2^2 q(\xi)\right)u(s, x_0)\right]_{s=-a_2x_0/a_1} \\ &\quad - 2a_1^2 a_2\left[\left(\frac{\partial^2}{\partial s^2} - a_1^2 q(\xi)\right)u(s, x_0)\right]_{s=-a_1x_0/a_2}.\end{aligned} \tag{32c}$$

In the case of an equation of the type (32a), for which $a_1 = a_2 = 1$, one obtains a reflection formula either directly from (31) by evaluating residues at double poles, or from (32c) by passing to the limit with a_1 and a_2. The resulting formula is

$$\xi_0^2 u(x_0, x) = \left[\left(\frac{\partial^2}{\partial s^2} + 2x_0\frac{\partial^3}{\partial s^3} - x_0^2\frac{\partial^4}{\partial s^4} + 2q(\xi) - 4x_0 q(\xi)\frac{\partial}{\partial s} + x_0^2 q(\xi)\frac{\partial^2}{\partial s^2}\right)u(s, x)\right]_{s=-x_0}.$$

Taking into account the initial conditions (32b) we can obtain a simpler formula for u itself, which does not involve any quadratures:

$$u(x_0, x) = \left[\left(-1 - 2x_0\frac{\partial}{\partial s} - x_0^2\frac{\partial^2}{\partial s^2} + x_0^2 q(\xi)\right)u(s, x)\right]_{s=-x_0}. \tag{33}$$

Formula (33) contains formula (3c) for the bi-harmonic equation as the special case $q = -\xi_1^2$. Reflection formulae for the more general polyharmonic case can be obtained from (31).

One observes that the reflection formulae for partial differential equations of the types (17b, c) have been obtained formally from the formula (19) which applies to ordinary differential equations, by treating $P(\xi_0, \xi)$ as an ordinary differential operator in ξ_0. For general forms $P(\xi_0, \xi)$ the *formal* expressions have no immediate concrete interpretation, since the formal roots ξ_0 of the equation $P(\xi_0, \xi)=0$ are only algebraic functions of differential operators. For equations of the special type (17b, c) meaningful formulae are obtained, due to the fact that in that case the quotients of roots are ordinary numbers instead of operators.

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