BOOK REVIEWS

The geometry of geodesics. By Herbert Busemann. New York, Academic Press, 1955. 10+422 pp. $9.00.

This book represents a most interesting and successful effort to treat in a unified fashion the geometry of a class of metric spaces general enough to include Riemannian and Finsler spaces. Only the barest minimum is postulated; in particular, the spaces are metric; bounded infinite sets have limit points; any two points can be connected by a (geodesic) segment, i.e. a curve whose length equals the distance of the points; every point has a neighborhood in which a geodesic segment can be prolonged in a unique way. It is a consequence that a given segment can be prolonged indefinitely in both directions to yield a geodesic, i.e. a curve which is locally a segment. These are more or less paraphrases, the five axioms themselves being stated in terms of the metric, and in very simple form. The spaces satisfying them are called G-spaces and are the object of study throughout the book. They include the complete Riemannian and Finsler manifolds. It should be pointed out however that it is not assumed that the spaces are differentiable, or even topological manifolds. In fact, it is an important unsolved question of the theory as to whether the axioms imply this. It is shown that the dimension, which may, so far as is known, be infinite, is the same at each point, and that should it be two, then the space is a manifold. It is a remarkable fact that in many instances it is shown that the addition of a single property or axiom is enough to characterize well-known geometries, of various types. For example if $B(x, y)$ denotes the bisector (set of points equidistant from) x and y, then the assumption that bisectors are flat, i.e. contain with each pair of points a segment joining them, implies that the space is euclidean, hyperbolic or spherical and of dimension greater than one. This implies a second characterization of these same spaces as solutions to the Helmholtz-Lie problem, to wit: if for any two isometric point triples of a G-space a motion exists which carries the first into the second, the same conclusion as above follows.

In all there are six chapters; the first deals with general concepts and contains as a highlight a solution in the large of the inverse problem of the calculus of variations for a system of curves in the plane: if each curve of the system goes to infinity at each end and any two points lie on exactly one curve, then the plane may be metrized as a G-space with these curves as geodesics. Chapter II is devoted to G-

spaces which can be imbedded topologically (but not isometrically) and with preservation of geodesics in projective spaces. These spaces the author calls Desarguesian. He shows that they can be characterized by two conditions: (i) the geodesic through two distinct points is unique and (ii) either the space is two dimensional and Desargues' theorem and its converse hold or it is a higher dimensional space and any three points lie on a plane. Special Desarguesian spaces are discussed, in particular Minkowski (or finite dimensional Banach) spaces which provide the local geometry in Finsler spaces in the same way as does euclidean space for Riemannian spaces. Chapter III deals with the theory of perpendiculars and parallels in a G-space. Definitions, too technical to repeat here, are given for these two notions and in terms of them a form of the axiom of parallels is stated, all of course, reducing to the usual notions in euclidean space. Elliptic, Minkowskian and euclidean geometries of dimension greater than two are characterized in this context.

Chapter IV is devoted to covering spaces of G-spaces, and using the theory developed there it is shown that in a G-space satisfying (i) above either all geodesics are straight lines or all are great circles of the same length. Theorems on the existence of closed geodesics, geodesics on tori, and on geodesic transitivity in G-spaces conclude the chapter. Chapter V, entitled *The influence of the sign of the curvature on geodesics* is of special interest in connection with the author's contention that most of the more geometric theorems on Riemannian spaces can be proved, by his methods, without differentiability assumptions and indeed without the assumption that the metric is Riemannian, so that they still hold, when suitably formulated, in the general class of spaces here considered. Sectional curvature is not defined as such, but simple geometric definitions of negative, positive, non-negative and nonpositive curvature are given for G-spaces, and it is shown that many classical theorems such as those of Hadamard for spaces of nonpositive curvature and the famous theorems of Cohn-Vossen for open surfaces hold equally well in G-spaces. To this end a generalized Gauss-Bonnet theorem is developed. The sixth and last chapter deals with homogeneous spaces and touches on questions related to the work of H. C. Wang and J. Tits on two point homogeneity (closely related to the Helmholtz-Lie problem mentioned above). Several examples are discussed in detail.

This is a book with a very original method of approach to problems of what might be called "general differential geometry" (with a slant toward the foundations of geometry). It assumes very little and proves a great many theorems, many of which were not known, certainly not in so general a form, before the work of the author. For this reason it is not a book that can be skimmed lightly, but rather it must be studied to be followed. The organization, however, is good and the proofs clearly written, their length in many instances being due to the extremely weak assumptions made. It must certainly be counted an important addition to the literature and will deserve the careful consideration of mathematicians interested in the geometry of Riemannian and, particularly, Finsler spaces.

WILLIAM M. BOOTHBY

Structure of a group and the structure of its lattice of subgroups. By Michio Suzuki. (Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge. Heft 10.) Berlin, Springer, 1956. 6+96 pp. DM 16.50.

The lattice of subgroups of a given group has been studied for a long time, even before lattice theory was recognized or named. The converse problem—what can be told about a group from knowledge of its lattice of subgroups—was first studied by Ada Rottländer in 1928, followed in the next few years by R. Baer and O. Ore. In the past fifteen years much more has been learned, through the efforts of a number of mathematicians including the author. The present monograph is the first collected presentation of this work. The author thoroughly surveys the known facts, rounding them out with additional results not previously published. Substantial familiarity with groups and lattices is assumed; thereafter the presentation is self-contained except for the omission of proofs or details (for which references are given) in the advanced stages of some developments.

Chapter I discusses groups which have special kinds of lattices of subgroups, such as distributive, modular, etc. In this respect, knowledge seems relatively weak in the case of complemented lattices. Chapter II considers the case of two groups G and H whose lattices of subgroups are isomorphic; then the author calls H a projectivity of G (more commonly G and H have been called lattice-isomorphic or structurally-isomorphic). This situation has been essentially completely characterized for finite groups, and largely so in the infinite case. One might add to the bibliography the recent paper of B. Jónsson [Mathematica Scandinavica, vol. 1 (1953) pp. 193–206]. Chapter III presents analogous results for lattice-homomorphic groups; for most of these results it is required that the homomorphism be complete (i.e., hold even for infinite joins and meets). Chapter IV considers cases where the lattices of subgroups are dually-isomorphic.

It is impressive to see how much is known about these matters as