SCIENTIFIC REPORT ON THE SECOND SUMMER INSTITUTE

SEVERAL COMPLEX VARIABLES

The Second Summer Institute, devoted to several complex variables, was held from June 21 to July 31, 1954 at the University of Colorado. It was supported by a grant from the National Science Foundation to the American Mathematical Society. This report outlines much of the work presented at the seminars at the Summer Institute. The report has been written under the supervision of an Editorial Committee consisting of S. S. Chern, W. T. Martin, and Oscar Zariski. The report is presented herewith in three parts, each of which has its own bibliography and may be read independently of the other two parts. Part I, which is based on the Analysis seminar, was prepared under the editorial supervision of W. T. Martin; Part II, Complex Manifolds, was written by S. S. Chern; and Part III, Algebraic Sheaf Theory, by Oscar Zariski. Detailed acknowledgments for each of the three parts of the report are given in the separate parts.

The report does not cover the work done on a seminar on Algebraic Geometry since the topic of that seminar—the problem of existence of minimal models for algebraic surfaces—will be the subject of a separate memoir by Oscar Zariski. Several lectures on the Albanese variety were given in that seminar by W. L. Chow.

PART I. REPORT ON THE ANALYSIS SEMINAR

Introduction. In the analysis seminar talks were given by several members of the Institute on topics ranging from the kernel function and automorphic functions to functions on abstract algebras and applications of function theory to linear partial differential equations. The talks included not only reports of work previously done in the field but also recent work done by the participants.

In this part we present a brief report of the seminar. This part is divided into sections corresponding to the lectures on a given topic. Some unsolved problems of interest have been included in the sections to which they are related. The material in each section is based upon a written summary prepared by the speaker or speakers as follows:

§§1A and 4: H. J. Bremermann;

§1B: M. Maschler;

§§2A and 6: S. Bergman;
§2B: R. C. Gunning;
§3A: W. L. Baily;
§3B: J. H. Sampson;
§5A: H. G. Haefeli;
§5B: N. S. Hawley.

In addition the Editorial Committee wishes to thank Doctors W. L. Baily and R. C. Gunning for their assistance in the planning and coordinating of the material of this part.

1. **Applications of kernel functions to some geometrical problems.** In this section we discuss some applications of the theory of the kernel function. A general treatment of the kernel function with further applications may be found in Bergman [**3**; **5**; **6**] and other papers cited there.

A. *Domains of holomorphy.* The Bergman kernel function $K_D(z, \bar{t})$ of a domain $D \subset C^n$ is a holomorphic function of (the $2n$ variables) $z, \bar{t}$ in the product domain $D_z \times D_{\bar{t}}$. The envelope of holomorphy E of a product domain is equal to the product of the envelopes of holomorphy of the domains: $E(D_z \times D_{\bar{t}}) = E(D)_z \times E(D)_{\bar{t}}$. Therefore for any domain D, $K_D(z, \bar{t})$ is holomorphic in $E(D)_z \times E(D)_{\bar{t}}$. In particular (letting $\bar{t} = \bar{z}$), $K_D(z, \bar{z})$ has a (plurisubharmonic) continuation into $E(D)$. Some of the properties of the kernel function continue to hold in the larger domain $E(D)$. For instance the reproducing property holds in the following sense: *Let $f(z)$ be holomorphic and let $f \in \mathcal{L}^2(D)$. Then $f(z) = \int_D K_D(z, \bar{t}) \overline{f(t)} d\omega_t$ not only for $z \in D$ but also for $z \in E(D)$.* (However the property $f \in \mathcal{L}^2(D)$ does not in general imply $f \in \mathcal{L}^2(E(D))$.) Also the Hermitian form

$$ds^2 = \sum_{\mu,\nu=1}^{n} \frac{\partial^2 K_D(z, \bar{z})}{\partial z_\mu \partial \bar{z}_\nu} dz_\mu d\bar{z}_\nu$$

is positive definite and invariant with respect to holomorphic transformations not only in D but also in $E(D)$.

We say that "$K_D(z, \bar{t})$ becomes infinite everywhere at the boundary of D" if for arbitrarily large real M the closure of $\{z \mid K_D(z, \bar{z}) < M\}$ is contained in D. Bremermann [**2**; **5**] has shown that a necessary condition for K to become infinite everywhere at the boundary of D is that D be a domain of holomorphy. He has given counter-examples to show that this is not a sufficient condition. However he has shown that any domain of holomorphy can be approximated by domains D_ν for which K does become infinite everywhere at the boundary. Each of the D_ν can be approximated by $\{z \mid K_{D_\nu}(z, \bar{z}) < M\}$, which turn out to be regions of holomorphy. Both approximations together give Bremermann's theorem [2]:

Any domain of holomorphy can be approximated from the interior by regions of holomorphy each having an infinitely differentiable boundary surface.

Bremermann has proposed the following two problems:

1. Express the kernel function for simple domains explicitly in terms of known transcendental functions.
2. Develop this theory for complex manifolds.

B. *Minimal and representative domains.* As there is no known analogue to the Riemann mapping theorem in the space of two complex variables, it is interesting to know that it is at least possible to map an arbitrary domain onto a domain which has certain properties. (In order to save space, the results of this portion are stated for the case of two complex variables. Similar theorems hold for the space of n complex variables, $n \geqq 1$.) In Bergman [5] (and other papers mentioned there) two kinds of such domains were introduced: *minimal domains* and *representative domains*. A minimal domain D in z_1, z_2-space with respect to a point $t \equiv (t_1, t_2) \in D$ as center is a domain which has the property that under any pseudo-conformal transformation $w_k = w_k(z_1, z_2)$, $(k=1, 2)$, $\partial(w_1, w_2)/\partial(z_1, z_2) = 1$ at $z=t$, its (four-dimensional) volume does not decrease. We allow also mappings in which w_k are not single-valued functions, provided that the Jacobian of the transformation is a single-valued function in D. In this case we identify the image points which correspond to the same point in D. A representative domain B with $r \in B$ as center is a domain which satisfies

$$z_j = M^{(j)}(z_1, z_2)/M(z_1, z_2) + r_j, \qquad j = 1, 2,$$

where M is the function which minimizes the integral $\int_B |f(z)|^2 d\omega$ under the conditions $f \in \mathcal{L}^2(B)$, $f(r) = 1$ ($d\omega$ is the volume element), and where $M^{(j)}$ is the function which minimizes the same integral under the conditions $f \in \mathcal{L}^2(B)$, $f(r) = 0$, $\partial f/\partial z_j = 1$, $\partial f/\partial z_i = 0$ $(i \neq j)$ at $z = r$. (We do not explicitly denote the dependence of the functions M, M' and M'' upon r and upon the domain B.)

A minimal or a representative domain does not have to be *schlicht* provided the point t does not lie on a branch manifold. Schiffer [1] has given sufficient conditions for a minimal domain to be schlicht. Very little is known about the geometric shape of minimal or representative domains; yet, since *any* domain which has a Bergman kernel function can be mapped pseudo-conformally onto them, knowledge of properties of these domains enables us to deduce various results which are of interest in the theory of pseudo-conformal transformations, Bergman [5], Maschler [1].

Maschler [1] has proved the following theorem.

A necessary and sufficient condition for a domain D to be a minimal domain with respect to a point t as center is that its Bergman kernel function satisfy a relation $K_D(z, \bar{t}) = \text{const.}$ for $z \in D$. The value of this constant is the reciprocal of the volume of D.

From this result Maschler has deduced other properties of minimal domains which give some information about their shape and also lead to distortion theorems. It can be shown that Reinhardt circular domains are both minimal and representative domains with their center at their center of gravity. Maschler has pointed out, however, that this phenomenon is not true in general. In fact he has obtained the following result, Maschler [2]: *A necessary and sufficient condition for a minimal domain D to be also a representative domain with the same center t is that $\partial K_D(z, \bar{t})/\partial \bar{t}_j = A_j(z_1 - t_1) + B_j(z_2 - t_2)$, $(j = 1, 2)$, for $z \in D$, where A_j and B_j are constants (A_1, B_2 are real; $A_2 = \bar{B}_1$ and $A_1B_2 - A_2B_1 \neq 0$). The kernel function of any domain Δ which can be mapped pseudo-conformally onto a domain D of this type such that the Jacobian of the transformation is regular and different from zero at τ (the inverse image of t in Δ) has the property that its kernel function satisfies the following differential equation:*

$$S(w; \tau) = \frac{1}{K^4} \begin{vmatrix} K & K_{\bar{1}} & K_{\bar{2}} \\ K_1 & K_{1\bar{1}} & K_{1\bar{2}} \\ K_2 & K_{2\bar{1}} & K_{2\bar{2}} \end{vmatrix} = \text{const.}$$

for $w \in \Delta$, where the elements of the determinant are defined as follows:

$$K = K_\Delta(w; \bar{\tau}), \qquad K_j = \partial K_\Delta(w; \bar{\tau})/\partial w_j,$$
$$K_{\bar{j}} = \partial K_\Delta(w; \bar{\omega})\partial \bar{\omega}_j \ \text{at} \ \omega = \tau$$

and

$$K_{j\bar{k}} = \partial^2 K_\Delta(w; \bar{\omega})/\partial w_j \partial \bar{\omega}_k) \ \text{at} \ \omega = \tau,$$
$$w = (w_1, w_2), \qquad \omega = (\omega_1, \omega_2).$$

This identity yields information about the kernel function for the case $w \neq \tau$. Since Bergman [3] has shown that the expression $S(w; w)$ is invariant under pseudo-conformal transformations, the value of the constant can be computed.

2. **Distinguished boundary surfaces.**

A. *Value distribution problems; examples.* When we attempt to generalize the methods of the theory of functions of one complex variable to the case of two variables the following class of domains plays an important role. On the (three-dimensional) boundary of a domain of this class lies a (not necessarily connected) two-dimensional surface $\mathfrak{F}^2$ such that every function regular in the closed domain assumes the maximum of its absolute value *not only on the three-dimensional boundary but even at a point of* $\mathfrak{F}^2$. (Superscripts denote the dimension of manifolds.) The surface $\mathfrak{F}^2$ is denoted as the distinguished boundary surface of the domain under consideration. For instance, if the domain is bounded by finitely many segments of analytic hypersurfaces, then the sum of intersections of these hypersurfaces forms the distinguished boundary surface $\mathfrak{F}^2$.

The Cauchy and Poisson formulas are important tools in investigating functions of one complex variable. As has been shown, Bergman [**1**; **2**; **12**]; Weil [**1**; **2**] there exists, for a large class of domains with distinguished boundary surfaces in several complex variables, an integral formula expressing the value of the function inside the domain in terms of its value on the distinguished boundary surface. This formula can be considered as a *generalization of the Cauchy formula* in one variable. However *it has the disadvantage that in the case of more than one variable its kernel depends on the domain.* Functions which are orthogonal when integrated over the distinguished boundary surface are introduced in Bergman [**3**; **13**; **16**]; these generalize the functions introduced by Szegö [**1**], which are orthogonal when integrated over the boundary curve of a domain in one variable. By using various positive weighting functions we then obtain various formulas expressing a function inside the domain in terms of its values on the distinguished boundary surface.

We proceed now to the question of analogs of the Poisson formula and of their applications, in particular, applications to the question of value distributions of functions of two variables. Let us note that the situation in this case differs in many respects from that in one variable. A function of two variables vanishes on segments of analytic surfaces. One of the problems of the theory of functions of two complex variables is to associate with these segments certain functionals which characterize some properties of these segments and which on the other hand are connected with the growth of the function. We are, in particular, interested in determining functionals which can be considered as generalizations of such notions as

$$D_\lambda(a, r) = \sum |z^{(\nu)}(a)|^{-\lambda},$$

where $z^{(\nu)}(a)$ are roots of $f(z) = a$ in the circle $|z| < r$. In the case of one variable derivations of many results in this direction are based on the possibility of solving boundary value problems for harmonic functions. In the case of a domain with a distinguished boundary surface, the boundary value problem for B-harmonic functions, (i.e., for the real parts of analytic functions of two complex variables) with arbitrary values prescribed on this surface does not always have a solution. In Bergman [1; 6; 16], Bergman and Schiffer [1] and Bremermann [5] various types of *functions of extended class* are introduced. Using these we can always solve the boundary value problem mentioned above and generalize potential-theoretic methods (in particular those of Nevanlinna and Ahlfors) to the case of two variables.

EXAMPLE 1. Consider the 2-parameter family of closed surfaces $\mathfrak{F}^2(r, s)$ in (z_1, z_2)-space defined by

$$\mathfrak{F}^2(r, s) = \{(z_1, z_2) \mid |z_2| = r, z_1 = h(z_2, t, \bar{t}), |t| = s\},$$

where $h(z_2, t, \bar{t})$ is a single-valued analytic function of $z_2, t, \bar{t}$. For any conveniently chosen real function $s = s(\rho)$, the set $\mathfrak{S}_r^3 = \bigcup_{r_0 \leq \rho \leq r} \mathfrak{F}^2(\rho, s(\rho))$ is a segment of the hypersurface $\mathfrak{S}^3 = \lim_{r \to \infty} \mathfrak{S}_r^3$. Then for an entire or meromorphic function $f(z_1, z_2)$ the set

$$\mathfrak{k}^1(a, r) = \mathfrak{S}_r^3 \cap \{(z_1, z_2) \mid f(z_1, z_2) = a\},$$

will be one-dimensional and will have a parametric representation

$$z_1 = z_1(\Psi, a), \qquad z_2 = z_2(\Psi, a), \qquad |z_2| < r$$

(with the parameter Ψ). The integral

$$B_\lambda(a, r) = \frac{1}{2\pi} \int_{\mathfrak{k}^1(a,r)} |z_1(\Psi, a)|^{-\lambda} d\Psi \tag{3}$$

can be considered as a generalization of the quantity $D_\lambda(a, r)$ in one variable. Generalizing the theorems of Hadamard and Borel it is possible to show that

$$\lim_{r \to \infty} B_\lambda(a, r)$$

exists if and only if λ is larger than a quantity connected with the growth of $f(z_1, z_2)$ on $\mathfrak{S}^3$. (See Bergman [14; 16], Bergman and Schiffer [1] and references to earlier papers of Bergman cited there.)

EXAMPLE 2. Since in pseudo-conformal transformations a pair of functions of two variables plays the same role as one function of one variable in conformal transformations, it is of interest to consider functionals connected with a pair of functions.

Let $\mathfrak{M}^4$ be a domain in (z_1, z_2)-space bounded by $a^3 \cup b^3$, where

$$\mathfrak{a}^3 = \{(z_1, z_2) \mid z_2 = \exp(i\lambda), 0 \leq \lambda \leq 2\pi\},$$
$$\mathfrak{b}^3 = \{(z_1, z_2) \mid z_1 = h(\lambda, z_2), 0 \leq \lambda \leq 2\pi, |z_2| \leq 1\},$$

and having distinguished boundary surface $\mathfrak{F}^2=\mathfrak{a}^3\cap\mathfrak{b}^3$. For each $|z_2^0|<1$ we form the function $w(z_1, z_2^0)$ mapping the set

$$\mathfrak{B}^2(z_2^0) = \mathfrak{M}^4 \cap \{(z_1, z_2) \mid z_2 = z_2^0\},$$

which is assumed to be simply connected, onto the unit disk. For a pair of analytic functions (f_1, f_2), let $P_1(z_2^0)$ and $P_2(z_2^0)$ be the products of the $C(\mathfrak{B}^2)$ distances between every zero of f_1 and every zero and pole respectively of f_2 on $B^2(z_2^0)$; here by the $C(\mathfrak{B}^2)$ distance between two points (z_1, z_2) and (z_1', z_2^0) is meant the quantity

$$\frac{w(z_1, z_2^0) - w(z_1', z_2^0)}{1 - w(z_1, z_2^0)\overline{w(z_1', z_2^0)}}\,.$$

Similarly $P_3(z_2^0)$ and $P_4(z_2^0)$ are defined by replacing the zeros of f_1 by poles of f_1.

The first functional to be considered is $\mathfrak{M}_1(\mathrm{P})$, the average of the *generalized Blaschke products*

$$\frac{P_1(z_2)P_4(z_2)}{P_2(z_2)P_3(z_2)}$$

over the circle $|z_2|<1$, from which have been removed circles of radius P around certain exceptional points (such as the projections of the points of intersection of pole and zero surfaces of f_1 with pole and zero surfaces of f_2). A second functional $\mathfrak{M}_2(\mathrm{P})$ is introduced similarly.

In Bergman [11] an upper bound is derived for $\mathfrak{M}_1(\mathrm{P})+\mathfrak{M}_2(\mathrm{P})$ in the following manner: we draw in M^4 tubes $t^4_{k\rho}$ of radius ρ around segments of zero-surfaces $\mathfrak{A}^2_{k1}=[z_1=\alpha_{k1}(z_2)]$ and pole-surfaces $\mathfrak{A}^2_{k2}=[z_1=\alpha_{k2}(z_2)]$ of f_k, $k=1, 2$. Upper bounds for $|(z_1-\alpha_{k\tau}(z_2))^{\tau}f_k|^{-1}$ and $|\partial[(z_1-\alpha_{k\tau}(z_2))^{\tau}f_k]/\partial z_1)|$ in $t^4_{k\rho}$ are denoted by B_k and D_k respectively, while A_k and C_k are the respective bounds for $|f_k|$ and $|\partial f_k/\partial z_1|$ in the complementary part $M^4-\sum t^4_{k\rho}$ of M^4. ($\tau=1$ if f_k has a pole, $\tau=-1$ if f_k has a zero.) The upper bound for $\mathfrak{M}_1(\mathrm{P})+\mathfrak{M}_2(\mathrm{P})$ is given in the form S_1+S_2 where S_1 depends essentially upon $\mathrm{P}, \rho, A_k, B_k, C_k, D_k, k=1, 2,$ the area of the above mentioned segments $\mathfrak{A}^2_{k\sigma}$ and the volume of M^4. S_2 depends essentially, in addition to the previously mentioned constants, also on the upper bounds L_k of $|(z_2-\gamma_{k2}(\lambda))\partial f_k/\partial z_2|$ in tubes of radius ρ around the intersection lines $[z_2=\gamma_{k2}(\lambda)]$ of b^3 with the surfaces $(\partial f_k/\partial z_2)=\infty$ and on the upper bounds M_k of $|\partial f_k/\partial z_2|$ in the complementary part of $\mathfrak{b}^3$. Further, S_2 depends also upon the length of the lines $\mathfrak{b}^3\cap[f_k=0]$, $\mathfrak{b}^3\cap[f_k=\infty]$.

EXAMPLE 3. See Bergman [7, pp. 170 ff.]. We note that in this case topological methods can be used, in particular Morse's theory of critical points.

B. *Poisson formulas in several variables.* The boundary value problem for the real parts of analytic functions of several complex variables (polyharmonic functions) in a domain D with a distinguished boundary surface B is in general insolvable. We may ask then for a polyharmonic function in D whose boundary values on B give a best approximation in L_2-norm to a prescribed set of values on B. A natural approach is to introduce a Bergman kernel function with respect to the distinguished boundary surface, Bergman [3]. However an approach based on Fourier series and integrals, Bochner [2], Bochner and Martin [1], will give an effective method of calculation in some special but interesting cases; only the results of this approach will be mentioned here.

Any polycylindrical domain may be transformed by a pseudo-conformal map onto a circular polycylinder

$$D: \ |z_j| < 1 \qquad (1 \leqq j \leqq n),$$

with distinguished boundary surface

$$B: \ |\zeta_j| = 1 \qquad (1 \leqq j \leqq n).$$

For any continuous distribution $u(\zeta_1, \cdots, \zeta_n)$ on B, Gunning has shown that the solution to this weak boundary value problem is given by:

$$u(z_1, \cdots, z_n) = \frac{1}{(2\pi)^n}\int_0^{2\pi} \cdots \int_0^{2\pi} u(\zeta_1, \cdots, \zeta_n) \cdot \left\{2\mathfrak{R}\left[\frac{\zeta_1 \cdots \zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}\right] - 1\right\} d\phi_1 \cdots \phi_n,$$

where $\zeta_j = e^{i\phi_j}$ $(1 \leqq j \leqq n)$.

The space of n^2 complex variables may be represented as the space of $n \times n$ complex matrices $z = (z_{ij})$; this representation imposes a ring structure on the argument z, in terms of which some of the approaches used in the case of functions of one variable may be extended to yield similar formulas for matrices. Replace the unit disk by the generalized unit sphere

$$D = \{z \mid z \cdot {}^t\bar{z} < I\}$$

where I is the unit matrix and the ordering $u < v$ means $v - u$ is positive definite. The distinguished boundary surface of D is the set

$$B = \{\zeta \mid \zeta \cdot {}^t\bar{\zeta} = I\},$$

the unitary group of $n \times n$ matrices. The equivalent of the angular measure $d\phi$ on the circle is the group-invariant measure $d\mu$ on the unitary group. Then for a continuous function $u(\zeta)$ on B vanishing at $-I$, Gunning has shown that the weak boundary value problem has the solution

$$u(z) = 2 \int_B u(\zeta) \cdot \Re \left[\frac{\det (I + z) \cdot \det (\zeta)}{\det (I + \zeta) \cdot \det (\zeta - z)} \right]^n d\mu.$$

3. **Projective models for fields of automorphic functions.**

A. *Mappings and imbeddings.* We would like to state, in a special case, some results on analytic mappings in several complex variables recently worked out in some degree of completeness by Cartan [**1**; **2**] (but which in principle go back at least as far as Osgood [**1**]) and to indicate their applications to a global imbedding theorem recently proved by Baily [**1**]. The problem in general with which these results deal is as follows:

Given analytic functions $f_1, \cdots, f_m$ at the origin a of the space of n complex variables C^n such that $f_1(a) = \cdots = f_m(a) = 0$ and a small neighborhood V of a, what does the image $\Phi(V)$ look like at the origin c of C^m, where $\Phi(z) = (f_1(z), \cdots, f_m(z))$? Osgood [**1**] proved a theorem which in part can be stated thus: *If $m = n$ and a is an isolated point of the set of common zeros of $f_1, \cdots, f_m$, then Φ is an open mapping in some neighborhood V of a, and with an appropriate choice of V, Φ represents V as a ramified N-fold covering of $\Phi(V)$, where N is some integer.* More recently Cartan [**1**; **2**] has obtained somewhat more general results which are formulated in terms of the notion of a germ of a general analytic space. Though space is lacking to develop the latter concept, part of Cartan's results can be stated as they apply to our problem thus:

If a is an isolated point of the common zeros of $f_1, \cdots, f_m$, then every holomorphic function g at a satisfies a distinguished polynomial relation of the form

$$g^N + a_1 g^{N-1} + \cdots + a_N = 0$$

in some small neighborhood of a, where $a_1, \cdots, a_N$ belong to the ring of holomorphic functions analytically generated by $f_1, \cdots, f_m$; moreover, $\Phi(V)$ is an irreducible analytic variety at a (in the sense of being the set of common zeros of analytic functions).

These results have an application to the problem: *If H is a finite subgroup of the general complex linear group in n variables, and V is a* *small neighborhood of the origin a of C^n stable under H, how may V/H be regarded as an analytic variety?* Cartan has shown, on the basis of results related to those stated previously, that if I is the ring of holomorphic functions at a invariant under H and $f_1, \cdots, f_m$ generate I analytically (so that a is automatically an isolated point of the set of their common zeros), then, for a sufficiently small neighborhood V of a, $\Phi(V)$ is an irreducible variety at $\Phi(a)$ and its local ring there is naturally isomorphic to I; since it is easily seen that I is integrally closed in its quotient field, this means that $\Phi(V)$ is analytically normal at $\Phi(a)$. This result has been used by Baily in examining the nature of the quotient space of an analytic manifold D by a discrete group G of analytic self-homeomorphisms in case D/G is compact and there exists a positive G-complex line bundle over D, Baily [**1**]. In this case he has shown that D/G can be regarded as an algebraic variety, which is locally analytically normal. A similar result has also been obtained by Cartan and Serre, see Cartan [**1**; **2**], in the special case when D is a bounded domain in C^n.

B. *Automorphic varieties.* Consider a bounded domain D in the space of n complex variables $z_1, \cdots, z_n$, and let Γ be a group of analytic homeomorphisms $\gamma: D \to D$. The elementary properties of Γ are established by means of Cauchy's integral theorem. (a) If we assume Γ is discontinuous at some point of D, then it is countable and is totally discontinuous in D. (b) Let the elements of Γ be γ_0, γ_1, γ_2, etc., where γ_0 is the identity, and let $J_\nu(z)$ denote the Jacobian

$$J_\nu(z) = \frac{\partial(\gamma_\nu z_1, \cdots, \gamma_\nu z_n)}{\partial(z_1, \cdots, z_n)} \qquad (\nu = 0, 1, 2, \cdots).$$

Then, under our assumption, the series

$$\sum_\nu |J_\nu(z)|^2$$

is majorized on every closed subset of D by a convergent series of constants.

Therefore, if $H(z)$ is bounded and holomorphic in D, the Poincaré θ-series

$$\vartheta(z) = \sum_0^\infty H(\gamma_\nu z) \cdot [J_\nu(z)]^k$$

represents a holomorphic function in D for any integral $k \geqq 2$. $\vartheta(z)$ satisfies the relation

$$\vartheta(\gamma_\mu z) = \vartheta(z) \cdot [J_\mu(z)]^{-k}.$$

A key result, Giraud [1] concerning θ-series is the following: If points $a_1, \cdots, a_r$ are selected in D, none of which is a fixed point of any γ in Γ other than γ_0, and if values for $\vartheta(z)$ and its partial derivatives of order $\leqq m$ (m fixed) are specified at the points $a_1, \cdots, a_r$, then for all suitably large weights k there exist θ-series $\vartheta(z)$ which, with their partial derivatives of orders $\leqq m$, assume at $a_1, \cdots, a_r$ the prescribed values within any prescribed error $\epsilon > 0$.

For simplicity we suppose now that Γ has no maps other than γ_0 which have fixed points in D. The space $M = D$ (mod Γ) can always be regarded as a complex manifold, and we further assume that Γ is such that M is compact.

Quotients of θ-series of the same weight k define meromorphic functions on M, and it is easily shown by the above result that the field $\mathcal{J}$ of meromorphic functions on M has transcendance degree n over the complex numbers. Moreover, the functions of $\mathcal{J}$ separate points of M. From the lemma, Sampson [1] has established that the dimension ρ_k of the linear space of holomorphic densities of weight k on M tends to infinity with k, and from this that M admits only finitely many analytic self-homeomorphisms. Hawley [1] obtains this result from his theory of Picard domains.

In an oral communication to Sampson, G. Washnitzer has indicated how the holomorphic densities on M can be used to obtain a nonsingular imbedding of M in complex projective space, whence, by Chow's theorem, Chow [1], M is algebraic.

4. Domains of holomorphy; plurisubharmonic functions and Oka's lemma.

1. A real valued function V is called *plurisubharmonic* in a domain $D \subset M^n$ (M^n a complex manifold) if and only if the following conditions hold: (a) $-\infty \leqq V < \infty$; (b) V is upper semi-continuous; and (c) $\partial^2 V(z+\lambda a)/\partial\lambda\partial\bar{\lambda} \geqq 0$ (defined as a distribution) holds for all $z+\lambda a \in U_i$ and all U_i, where $\{U_i\}$ is a covering of D, the z's are local coordinates in U_i, λ is a complex number, and $a \equiv (a_1, \cdots, a_n) \in C^n$ (the space of n complex variables).

There are equivalent definitions (see Lelong [1; 4] and Bremermann [1]). In a *schlicht* space condition (c) means that the function V is subharmonic in the intersection of any one-dimensional analytic plane with D. This definition can be used for Banach spaces, see Bremermann [8]. The plurisubharmonic functions are a proper subclass of the subharmonic functions of $2n$ real variables for $n > 1$; for $n = 1$ they coincide.

2. The functions built up from $\log |f|$, f holomorphic in D, by addition, multiplication with positive (real) numbers, taking upper

envelopes, and a closure operation are called *Hartogs functions*, Bochner-Martin [1]. Every (upper semi-continuous) Hartogs function is a plurisubharmonic function. If D is a domain of holomorphy the converse is true. This is a consequence of Oka's lemma (see Bremermann [7]). Bremermann has shown that the converse is not true if D is not a domain of holomorphy. In this case every Hartogs function has a "Hartogs continuation" into the envelope of holomorphy of D, but by the use of tube domains, Bochner [1], Bremermann has shown that plurisubharmonic functions can be constructed which do not possess a plurisubharmonic continuation, Bremermann [6]. This disproves the conjecture by Bochner-Martin [1] that the plurisubharmonic functions and the Hartogs functions coincide. This conjecture has been investigated by Lelong [2] and Hitotumatu [1].

3. Let $P=\{z \mid z\in D\wedge \log|f_1(z)|<0\wedge \cdots \wedge \log|f_k(z)|<0\}$, where $f_1, \cdots, f_k$ are holomorphic in D. Let the closure $\overline{P}$ of P be contained in D, $\overline{P}\subset D$. Then P is called an analytic polyhedron, and $B=\{z \mid z\in\overline{P}\wedge\log|f_1(z)|=0\wedge\cdots\wedge\log|f_k(z)|=0\}$ its distinguished boundary surface. Given continuous boundary values on B, take the class of all functions plurisubharmonic in P and less than or equal to the given boundary values on B. Bremermann has shown that the envelope of this class exists, is plurisubharmonic, and assumes the boundary values. The upper envelope function is different from Bergman's "function of the extended class" for the same domain P but serves the same purpose, Bergman [6] (and references cited there); Bremermann [7].

4. Plurisubharmonic functions generate metric forms since $\sum_{\mu,\nu=1}^{n}(\partial^2 V/\partial\bar z_\mu\partial z_\nu)dz_\mu d\bar z_\nu$ (defined as current) is positive semidefinite. Also several exterior differential forms are connected with plurisubharmonic functions, Lelong [4]. In particular: Let W^p be an analytic set in D of pure complex dimension p. Let $\{U_k\}$ be a covering of D. In each U_k, W^p can be represented as the intersection of the zero manifolds of $n-p$ functions f_s^k holomorphic in U_k. Let $V_s=\log|f_s^k|$ and $dzd\bar z V_s=\sum_{\mu,\nu=1}^{n}(\partial^2 V/\partial z_\mu\partial\bar z_\nu)dz_\mu\wedge d\bar z_\nu$. Then the current $\Theta_k=(i/\pi)^{n-p}dzd\bar z V_1\wedge\cdots\wedge dzd\bar z V_{n-p}$ is defined in U_k. Then $\Theta=\Theta_k$ defines a closed current in D. For the integral over W^p of a p-form with compact support in D we have $\Theta(\phi)=\int_{W^p}\phi$, Lelong [4].

5. Given a domain D in a metric space, then we define:

$$\delta_D(z) = \sup r \ni \{z' \mid |z'-z| < r\} \subset D;$$

$\delta_D(z)$ is the distance of the point z from the boundary of D. $\delta_D(z)$ is continuous for any domain. A domain D in a complex Banach space is called "pseudo-convex" if and only if $-\log\delta_D(z)$ is plurisubharmonic in D. The definition of convex functions can be made completely analogous to the definition of the plurisubharmonic functions. (We could call convex functions for one real variable "sublinear" and for several real variables "plurisublinear.") Bremermann [5] has proved that a domain is convex if and only if $-\log \delta_D(x)$ is a convex function in D, and he [8] has proved the same result for real Banach spaces. "Complex convexity" (as we denote pseudo-convex domains and plurisubharmonic functions together) is the extension of the notion of convexity from real to complex spaces. The theories can be developed in parallel. One example: Let S_ν, S_0 be domains on one-dimensional analytic surfaces, T_ν, T_0 the boundaries, and $\lim_{\nu\to\infty} T_\nu = T_0$ and $\lim_{\nu\to\infty} S_\nu = S_0$. If for any such sequence for which S_ν, T_ν and $T_0 \subset D$ also $S_0 \subset D$, then we say "the theorem of continuity holds for D." D is pseudo-convex if and only if the theorem of continuity holds for D. (Bremermann [1], Lelong [3].) From this follows: D is pseudo-convex if and only if a function V, plurisubharmonic in D, exists such that for all real M the closure of $\{z \mid V(z) < M\}$ is contained in D, in other words if and only if a plurisubharmonic function exists that becomes infinite everywhere at the boundary of D. The same theorem holds for convex domains, if we replace "one-dimensional analytic surfaces" by straight lines, and the plurisubharmonic functions by convex functions.

6. A domain is called a "tube" domain if and only if it is of the form $\{z \mid x \in B,\ y \text{ arbitrary}\}$ (x real part, y imaginary part), Bochner [1]; Bochner-Martin [1]. A tube domain is pseudo-convex if and only if B is convex. A function defined in a tube domain and not depending on the imaginary parts is plurisubharmonic if and only if its restriction to B is convex in B. Therefore one obtains for every theorem on plurisubharmonic functions and pseudo-convex domains (that does not involve "existence") by specialization to tube domains a corresponding theorem on convex functions and domains. The converse is not true, of course. However, to every convexity theorem we have the problem: does the corresponding theorem on complex convexity hold? Bremermann [5].

7. Oka's lemma states: A schlicht and finite domain is a domain of holomorphy if and only if it is a pseudo-convex domain. The lemma solves a problem established 1910 by E. E. Levi [1] (compare Behnke-Thullen [1]; and Behnke-Stein [2]). The lemma was proved for two variables by Oka [2] and for n variables by Norguet [1] and Bremermann [4]. Recently a new proof for arbitrary n and for a certain class of complex manifolds was given by Oka [5]. The proof of the main part "if pseudo-convex, then domain of holomorphy" is

too involved to be sketched here. The other part can be proved in an easy way from the theorem: Let D be a domain of holomorphy and S, T sets such that $\overline{S\cup T}\subset D$, S, open and for any function $f(z)$ holomorphic in D assume that $\sup_{z\in S\cup T}|f(z)| = \sup_{z\in T}|f(z)|$. Let $g(z)$ be holomorphic in D. Then $\inf_{z\in S\cup T}\ \delta_D(z)|e^{g(z)}| = \inf_{z\in T}\ \delta_D(z)|e^{g(z)}|$. Also the holomorph-convexity (in the sense of Cartan-Thullen [1]) follows from this theorem immediately with $g(z)\equiv 0$, Bremermann [5].

PROBLEMS: 1. To find a substitute for $\delta_D(z)$ on complex manifolds.

2. To study pseudo-convex domains on complex manifolds. As a definition the following could be used: A domain D is called pseudo-convex if and only if there exists a function V, plurisubharmonic in D, such that the closure of $\{z|\ V(z)<M\}$ is contained in D for all real M.

3. Prove Oka's lemma for arbitrary complex manifolds.

4. Study the sheaf of germs of plurisubharmonic functions.

5. Functions on algebras.

A. *Functions on a Clifford algebra.* This is a report on functions of a hypercomplex variable, extensive work on which has been carried out by R. Fueter [1] at the University of Zurich, and his school including H. G. Haefeli [1] in this country.

Let C be a Clifford Algebra of order 2^{n-1} with the basis

$$e_1, \cdots e_{n-1}$$

and the relations

$$e_k^2 = -1 \text{ (elliptic case)} \tag{1}$$

and

$$e_h e_k = -e_k e_h \text{ for } h \neq k, \tag{2}$$

and let $\mathcal{L}$ be a module in C with the basis $e_0, e_1, \cdots, e_{n-1}$, where $e_0 = 1$ denotes the principal unit. A variable $x\in\mathcal{L}$ is defined $x=\sum_{k=0}^{n-1} x_k e_k$, where the x's are real variables, and a function $w\in\mathcal{L}$ as a mapping from $\mathcal{L}$ into itself $w=\sum_{k=0}^{n-1} u_k e_k$, where the u's are real functions of the n variables. One requires that all these functions are defined and possess continuous first partial derivatives in all variables in a region R of the n-dimensional euclidean space E^n. If D denotes

$$D = \sum_{k=0}^{n-1} \frac{\partial}{\partial x_k} e_k, \tag{3}$$

a function w is called right- or left-analytic in R, if

$$wD = 0 \text{ or } Dw = 0 \text{ holds in } R. \tag{4}$$

These are two systems of linear partial differential equations and they can be considered as extensions of the Cauchy-Riemann equations; they are equivalent to $(n^2-n+2)/2$ real conditions. This choice of definition of analyticity is not a formal one, but originates from the desire to generalize Cauchy's integral theorem. Indeed, if $R_1 \subset\subset R$, the integral

$$\int_{BdR_1} wdX = 0 \quad \text{if and only if} \quad wD = 0. \tag{5}$$

Let $\overline{D}=\sum_{k=0}^{n-1} (\partial/\partial x_k)\bar{e}_k$ with $\bar{e}_0=e_0$ and $\bar{e}_k=-e_k$ for $k\neq 0$; then $D\overline{D}=\Delta$ is the Laplacian in n variables and $wD=0$ implies

$$\Delta w = 0 \tag{6}$$

and this implies $\Delta u=0$. The components of w are again harmonic functions as in the classical case. Analogously to Cauchy's formula it is also possible to express the value of w at a point $x\in R$, by means of the values of w on the boundary of R_1:

$$w(x) = \text{const.} \int_{BdR_1} w(\xi)d\Xi \frac{\bar{\xi}-\bar{x}}{n(\xi - x)^{n/2}} \cdot \tag{7}$$

From this it is possible for even n to obtain a development of an analytic function around a regular point in a series of orthogonal, homogeneous polynomials.

Instead of the conditions (1) one can take

$$\begin{aligned} \overset{2}{e_k} &= 1 \quad \text{for } k = 1, \cdots, m \text{ and} \\ \overset{2}{e_k} &= -1 \text{ for } k = m+1, \cdots, n-1. \end{aligned} \tag{1'}$$

Analytic functions are defined as in (4), and (5) holds; but (6) is now a hyperbolic partial differential equation of second order and (7) becomes an integral equation, which has been solved for special cases with the *partie finie methode*.

One may take the variable x and the function w in two different modules or algebras, as long as the second is invariant by left multiplication with elements from the first. $x\in\mathcal{L}_1$ and $w\in\mathcal{L}_2$, $\mathcal{L}_1\mathcal{L}_2=\mathcal{L}_2$ and $D\in\mathcal{L}_1$. For particular choice of $\mathcal{L}_1$ and $\mathcal{L}_2$, (4) gives the Dirac equations and (6) the wave equation; (7) reduces to an integration over the intersection of BdR_1 with the characteristic cone through x, and the resulting integral equation has been solved.

If one takes instead of $\mathcal{L}$ a product system $P=[1, e_n][1, e_1, \cdots, e_{n-1}]$ and identifies $e_n=i$ with the ordinary complex unit, one obtains a class of analytic functions, which contains as special subclass the complex analytic functions

$$w = f_0 + f_1e_1 + \cdots f_{n-1}e_{n-1}, \tag{8}$$

where $f_k(z_1, z_2, \cdots, z_n)$ are analytic functions of n-complex variables. (5) holds again and gives for the subclass the first rigorous proof of Hartogs' theorem on analytic continuation of f_k. (7) holds only for the subclass, from which one concludes that f_k can be approximated with rational functions in P.

For $n=2$ one obtains the algebra of the quaternions. (5), (6) and (7) and Runge's approximation theorem hold without exceptions. Around an isolated singularity there exists a Laurent expansion and one defines nonessential and essential singularities. The first need not be poles and one has essential singularities along curves and surfaces, in the neighborhood of which the function can be expressed with the help of a Stieltjes integral. Analytic continuation and first attempts to define generalized Riemann surfaces have been made. Also quadruply periodic functions and applications to the theory of numbers have been studied.

PROBLEMS: 1. For what Clifford Algebras does (7) hold without restrictions? Conjecture of Fueter: If the absolute term in the characteristic equation has no zero divisor.

2. What are the different systems of linear partial differential equations (4) which imply the same partial differential equation of second order (6)? (elliptic and hyperbolic case).

3. When does there exist for a system of linear partial differential equations an algebra such that (4) implies (5)?

4. Can every *schlicht* region be the domain of definition for an analytic quaternion function?

5. Is a product representation of an analytic quaternion function possible?

6. What is the behavior of an analytic quaternion function around an isolated essential singularity?

B. *Functions on a complex algebra.* Let E denote the algebra defined on pairs (x_1, x_2) of complex numbers by the multiplication rule $(x_1, x_2)\cdot(y_1, y_2)=(x_1y_1-x_2y_2, x_1y_2+x_2y_1)$ and the addition rule $(x_1, x_2)+(y_1, y_2)=(x_1+y_1, x_2+y_2)$. This algebra is commutative, associative, and has no nilpotent elements. Hawley has developed a theory of E-valued functions on E which very closely parallels the classical theory of functions of one complex variable.

This E-function theory has Cauchy-Riemann equations, a Cauchy's theorem, and formula, calculus of residues, Taylor's and Laurent's expansions, etc. Let $X=(x_1, x_2)$ and $Y=(y_1, y_2)$, then the functions $Y=f(X)$ considered are those such that $y_1=u_1(x_1, x_2)$, $y_2=u_2(x_1, x_2)$ satisfy $\partial u_1/\partial x_1=\partial u_2/\partial x_2$ and $\partial u_2/\partial x_1=-\partial u_1/\partial x_2$, i.e. a very special class of mappings in two complex variables. The property of being an E-function, or E-analytic, is *not* invariant under arbitrary pseudo-conformal mappings. This last fact ultimately proves to be of great advantage, bringing within the scope of the E-theory a much larger class of analytic mappings than is apparent at first. A proper change of coordinates may reduce a quite general analytic mapping to an E-function. Partial differential equations (but not a unique set of them) for the new coordinates can be given. Of course, all of this is quite local.

Hawley has also developed a theory of conformal mapping in two complex variables on the basis of this E-function theory. Thus if C_1 and C_2 are two complex curves, they have a *complex* angle between them which is preserved by elements of E-function theory just as the real angle is preserved in ordinary conformal mappings.

Of course, the parallelism of the E-theory to the classical theory becomes less satisfactory the deeper we go. For the most part, analogies to existence theorems and boundary properties break down. Its strength lies in special cases, not generality.

Analogues to the theory of algebraic curves have been developed and one obtains a special class of algebraic surfaces for which one has an extended set of methods for investigating.

There are certain obvious extensions to more than two complex variables, but these will not be discussed here.

PROBLEM: "Characterise" a complex manifold which is complex analytically homeomorphic to a bounded domain in a space of several complex variables.

As a special case of this problem, consider the simply connected Picard domains. (For definition see Hawley [1, p. 638].) The Picard domains are a generalization from one to several variables of the domains possessing at least two boundary points. In the case of one variable it is trivial to prove that a simply connected Picard domain is analytically homeomorphic to a bounded domain. Is this proposition true for more variables?

6. Operators transforming functions of complex variables into solutions of linear partial differential equations. By the relation $\Psi=[g(z)+\bar{g}(\bar{z})]/2$ the linear space of harmonic functions Ψ is mapped onto the algebra of analytic functions of one complex variable. Here z and $\bar{z}$ are complex conjugates, $z=x+iy$, $\bar{z}=x-iy$, and x and y are real. If the function Ψ is continued to complex values of x, y the variables z, $\bar{z}$ become two independent complex variables. If we consider the harmonic function Ψ in a neighborhood of a given point P with coordinates $z_0 = x_0 + iy_0$, then the associated analytic function g satisfies the relation $g(z) = 2\Psi(z, \bar{z}_0) - \bar{g}(\bar{z}_0)$, and thus we note that the associate g changes by a constant when the point P varies.

Bergman has generalized this mapping to the case of linear partial differential equations

$$L(\Psi) \equiv \Psi_{z\bar{z}} + a\Psi_z + \bar{a}\Psi_{\bar{z}} + c\Psi = 0 \tag{1}$$

whose coefficients are entire functions. In this case we have the relation

$$\Psi = [p(g) + \bar{p}(\bar{g})]/2 \tag{2}$$

where $p(g) \equiv p[g(z), P]$ is Bergman's integral operator of the first kind. As in the harmonic case, the function g which we associate with a given solution Ψ of (1) depends upon the point P in the neighborhood of which the relation (2) is defined. This relation is initially defined in a sufficiently small neighborhood of the point P. Bergman has shown however that it can be continued so that it holds in the large. For example, it holds in the domain of regularity of g if this domain is simply connected.

If P is the origin 0 the functional $p(g)$ of (2) can be expressed in the form

$$\begin{aligned} &p[g(z), 0] \\ &= \exp\left(-\int_0^{\bar{z}} a d\bar{z}\right)\left[g(z) + \sum_{n=1}^{\infty} T^{(n)}(z, \bar{z}) \int_0^z (z - \zeta)^{n-1} g(\zeta) d\zeta\right] \end{aligned} \tag{3}$$

where the $T^{(n)}$ are functions of z, $\bar{z}$ which *depend only* on equation (1) and possess the property that $T^{(n)}(z, 0) = 0$. Bergman [**4**; **9**]. It should be noted, however, that due to the fact that integrals appear in (3) certain complications arise when the solutions Ψ defined in the small are continued in the large.

In this theory it is of basic importance that *various relations between g and Ψ are independent of the coefficients a, c* occurring in the operator L, or that various other relations depend only upon certain properties of these coefficients. As an example for this fact one sees from (2) and (3) that the relation

$$\Psi(z, 0) = \frac{1}{2}\left[g(z) + \exp\left(-\int_0^z \bar{a}(z, 0) dz\right) \bar{g}(0)\right]$$

eolds.

The use of the integral operator of the first kind shows that *most* *of the results of the theory of functions of one complex variable can be interpreted as theorems on real solutions of the differential equation* (1) (and not merely as theorems on harmonic functions). A survey of some of the results in this direction (with bibliographical data) can be found in Bergman [8, p. 38 ff.], and in Bergman [9] where multi-valued solutions Ψ are considered.

The theory of integral operators has been generalized in two directions: (1) the case when the coefficients a, c have singularities of certain types; and (2) the case of harmonic and more general equations in three variables as well as the case of certain systems of linear partial differential equations. See Bergman [9; 10]. As Bergman has pointed out, while investigations of flows of incompressible fluids lead to the theory of harmonic functions of two variables, the case of compressible fluids, when the hodograph method is used, introduces problems in the theory of a class of equations of the type described in (1). Further, as Bergman [14] and Kreysig [1] pointed out, special integral operators are useful for characterizing solutions of partial differential equations using the theory of ordinary differential equations.

In the remainder of this section we consider only the case (2). It seems natural that harmonic functions of three variables have to be mapped onto analytic functions of *two* complex variables. Bergman (see [4; 9; 10] and earlier references cited there) has introduced the operator

$$\begin{aligned} H(X) &= C_3(g, P) \\ &\equiv \frac{1}{\pi i}\int_{|\zeta|=1}\int_0^1 u^{1/2}\, \frac{d[u^{1/2}g(u\zeta^{-1}T^2, u\zeta(1-T)^2)]}{du}\, dT\, \frac{d\zeta}{\zeta}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} X &= (x, Z, Z^*) \equiv (x, (z+iy)/2, -(z-iy)/2), \\ u &= x + Z\zeta + Z^*\zeta^{-1}, \end{aligned} \tag{5}$$

and he has shown that (4) *transforms analytic functions g of two complex variables into harmonic functions $H(X)$ of three variables x, y, z.* He has further considered, Bergman [10], integral operators

$$P_3(G, P) \equiv G(X) + \int_0^1 M(r^2, \sigma^2)G(X\sigma^2)d\sigma$$

which transform harmonic functions $G(X)$ of three variables into solutions of the equation

$$\Lambda(\Psi) \equiv \Delta_3\Psi + C\Psi = 0$$

where C is an entire function of $r^2=x^2+y^2+z^2\equiv x^2-4ZZ^*$. The inverse of $p_3[C_3(g, P)]$, $g(Z, Z^*)=\Psi[2(ZZ^*)^{1/2}, Z, Z^*]$, is *independent of the coefficient* C of the equation $\Lambda(\Psi)=0$. In Bergman [9] solutions of $\Lambda(\Psi)=0$ are investigated, whose associates $g=g_1+(ZZ^*)^{1/2}g_2$ have the property that g_1, g_2 are rational or algebraic functions of Z, Z^*. In his lectures Bergman showed that various theorems in the theory of functions of two complex variables can be interpreted as theorems on solutions of $\Lambda(\Psi)=0$, theorems on more general differential equations in three variables, theorems on harmonic vectors, Bergman [9], and theorems on certain differential equations in four variables.

In this section we have discussed only very special integral operators, mapping functions of complex variables into solutions of linear differential equations and systems of such equations. To every differential equation there exist infinitely many such operators, and the question of determining and classifying them is of great interest. In particular the investigation of how different operators permit us to use the theory of analytic functions for the study of solutions of differential equations (for real and complex values of the arguments) represents an interesting task.

References

W. L. Baily

1. *On the quotient of an analytic manifold by a group of analytic homeomorphisms*, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 804–808.

H. Behnke and K. Stein

1. *Konvergente Folgen nichtschlichter Regularitaetsbereiche*, Annali di Matematica Pura ed Applicata (4) vol. 28 (1949) pp. 317–326.
2. *Die Singularitaeten der analytischen Funktionen mehrerer Veraenderlichen*, Nieuw Archief voor Wiskunde, Amsterdam, 1951, pp. 227–242.

H. Behnke and P. Thullen

1. *Theorie der Funktionen mehrerer komplexer Veraenderlichen*, Ergebnisse der Mathematik, vol. 3, 1934, no. 3.

S. Bergman

1. *Über ausgezeichnete Randflächen in der Theorie der Funktionen von zwei komplexen Veränderlichen*, Math. Ann. vol. 105 (1931) pp. 611–636.
2. *Über eine Integraldarstellung von Funktionen zweier komplexer Veranderlichen*, Rec. Math. (Mat. Sbornik) N.S. vol. 1 (43) (1936) pp. 851–862.
3. *Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la theorie des fonctions analytiques*, Interscience, 1941, and Memorial des Sciences Mathématiques vol. 106, Paris, 1947.
4. *Linear operators in the theory of partial differential equations*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155.
5. *Sur la fonction-noyau d'un domaine et ses applications dans la theorie des transformations pseudo-conformes*, Memorial des Sciences Mathématiques, vol. 108, Paris, 1948.
6. *The kernel function and conformal mapping*, Mathematical Surveys, no. 5, Amer. Math. Soc., 1950.

7. *Geometric and potential-theoretical methods in the theory of functions of several complex variables*, Proceedings of the International Congress of Mathematicians, vol. 2, 1950, pp. 165–173.
8. *Operatorenmethoden in der Gasdynamik*, Zeitschrift für angewandte Mathematik und Mechanik vol. 32 (1952) pp. 33–45.
9. *The coefficient problem in the theory of linear partial differential equations*, Trans. Amer. Math. Soc. vol. 73 (1952) pp. 1–34.
10. *Essential singularities of solutions of a class of linear partial differential equations in three variables*, Journal of Rational Mechanics and Analysis vol. 3 (1954) pp. 539–560.
11. *On zero and pole surfaces of functions of two complex variables*, Trans. Amer. Math. Soc. (1955) vol. 77 (1954) pp. 413–453.
12. *Zur Veranschaulichung der Kreiskörper und Bereiche mit ausgezeichneter Randfläche*, Jber. Deutschen Math. Verein. vol. 42 (1932–1933) pp. 238–252.
13. *Bounds for analytic functions in domains with a distinguished boundary surface*, Math. Zeit. vol. 63 (1955) pp. 173–194.
14. *Zur Theorie der Funktionen, die eine lineare partielle Differentialgleichung befriedigen*, Rec. Math. (Mat. Sbornik) N.S. vol. 2 (1937) pp. 1169–1198.
15. *Ueber meromorphe Funktionen von zwei komplexen Veranderlichen*, Compositio Math. vol. 6 (1939) pp. 305–335.
16. *Kernel function and extended classes in the theory of functions of complex variables*, Colloque sur les fonctions de plusieurs variables, Paris, Masson, 1953, pp. 135–157.

S. Bergman and M. Schiffer

1. *Potential-theoretic methods in the theory of functions of two complex variables*, Compositio Math. vol. 10 (1952) pp. 213–240.

S. Bochner

1. *A theorem on analytic continuation of functions in several variables*, Ann. of Math. vol. 39 (1938) pp. 14–19.
2. *Group invariance of Cauchy's formula in several variables*, Ann. of Math. vol. 45 (1944) pp. 686–706.

S. Bochner and W. T. Martin

1. *Several complex variables*, Princeton, 1948.

H. J. Bremermann

1. *Die Charakterisierung von Regularitaetsgebieten durch pseudokonvexe Funktionen*, Schriftenreihe des Math. Inst., Muenster, no. 5, 1951.
2. *Holomorphic continuation of the Bergman kernel function and the invariant metric in several complex variables*, Lectures on functions of a complex variable, . . . (Report of the Ann Arbor, Michigan 1953 conference.)
3. *Die Holomorphiehuellen der Tuben-und Halbtubengebiete*, Math. Ann. vol. 127 (1954) pp. 406–423.
4. *Ueber die Aequivalenz der pseudokonvexen Gebiete und der Holomorphiegebiete im Raum von n komplexen Veraenderlichen*, Math. Ann. vol. 128 (1954) pp. 63–91.
5. *Complex convexity*, Technical Report 38 (ONR), Contract 106-5, Stanford, California, 1954.
6. *On a conjecture on Hartogs functions*, To appear.
7. *A plurisubharmonic extended class of functions*, To appear.
8. *Complex convexity in Banach spaces*, To appear.

H. Cartan

1. *Séminaire Ecole Normale Supérieure*, 1951–1952.

2. *Séminaire Ecole Normale Supérieure*, 1953–1954.

H. CARTAN and P. THULLEN

1. *Zur Theorie der Singularitäten der Funktionen mehrerer Veränderlichen: Regularitätens-und Konvergengbereiche*, Math. Ann. vol. 106 (1932) pp. 617–647.

W. L. CHOW

1. *On compact complex analytic varieties*, Amer. J. Math. vol. 71 (1949) pp. 893–914.

R. FUETER

A list of the publications of Fueter and his students can be found at the end of the paper by H. G. Haefeli listed below.

G. GIRAUD

1. *Leçons sur les fonctions automorphes*, Paris, Gauthier-Villars, 1926, pp. 19–25.

H. G. HAEFELI

1. *Hyperkomplexe Differentiale*, Comment. Math. Helv. vol. 20 (1947) pp. 382–420.

NEWTON S. HAWLEY

1. *A theorem on compact complex manifolds*, Ann. of Math. vol. 52 (1950) pp. 637–641.

S. HITOTUMATU

1. *On some conjectures concerning pseudo-convex domains*, J. Math. Soc. Japan vol. 6 (1954) pp. 177–195.

E. KREYSZIG

1. *Integral operators in the theory of partial differential equations*, Journal of Rational Mechanics and Analysis vol. 4 (1955) pp. 907–923.

P. LELONG

1. *Les fonctions plurisousharmoniques*, Ann. École Norm. vol. 62 (1945) pp. 301–345.
2. *La convexité et les fonctions analytiques de plusieurs variables complexes*, J. Math. vol. 31 (1952) pp. 191–219.
3. *Domaines convexes par rapport aux fonctions plurisousharmoniques*, Journal d'Analyse Mathématique vol. 2 (1952–1953) pp. 178–208.
4. *Sur les dérivées d'une fonction plurisousharmonique*, C. R. Acad. Sci. Paris (1954) pp. 2276–2278.

E. E. LEVI

1. *Studii sui punti singolari essenziali delle funzioni analitiche di due o piu variabili complesse*, Annali di Matematica Pura ed Applicata (3) vol. 17 (1910) pp. 61–87.

M. MASCHLER

1. *Properties of minimal domains*, Proceedings of the International Mathematical Congress, Amsterdam, 1954.
2. *Minimal domains and their Bergman kernel function*, to appear in Pacific Journal of Mathematics.

F. NORGUET

1. *Sur les domaines d'holomorphie des fonctions uniformes de plusieurs variables complexes (passage du local au global)*, Bull. Soc. Math. France vol. 82 (1954) pp. 137–159.

K. OKA

1. *Sur les fonctions analytiques de plusieurs variables.* II. *Domaines d'holomorphie*, Journal of Science of the Hiroshima University Ser. A vol. 7 (1937) pp. 115–130.
2. VI. *Domaines pseudoconvexes*, Tôhoku Math. J. vol. 49 (1942) pp. 15–52.

3. VII. *Sur quelques notions arithmétiques*, Bull. Soc. Math. France vol. 78 (1950) pp. 1–27.
4. VIII. *Lemme fondamentale*, J. Math. Soc. Japan vol. 3 (1951) pp. 204–212, 259–279.
5. IX. *Domaines finis sans point critique interieur*, Jap. J. of Math. vol. 23 (1953) pp. 97–155.

W. F. Osgood
1. *Lehrbuch der Funktionentheorie*, 2d ed., vol. 2, part 1, Leipzig, Teubner, 1929.

J. H. Sampson
1. *A note on automorphic varieties*, Proc. Nat. Acad. Sci. U.S.A. vol. 38 (1952) pp. 895–898.

M. Schiffer
1. *Sur les domaines minima dans la theorie des transformations pseudo-conformes*, C.R. Acad. Sci. Paris vol. 207 (1938) pp. 112–115.

G. Szegö
1. *Über orthogonale Polynome, die zu einer gegebenen Kurve der Komplexen Ebene gehören*, Math. Zeit. vol. 2 (1921) pp. 218–270.

A. Weil
1. *Sur les series de polynômes de deux variables complexes*, C. R. Acad. Sci. Paris vol. 194 (1932) p. 1304.
2. *L'integral de Cauchy et les fonctions de plusieurs variables*, Math. Ann. vol. 111 (1935) pp. 178–182.

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Part II. Complex manifolds[1]

The notion of a complex manifold is a natural outgrowth of that of a differentiable manifold. Its importance lies to a large extent in the fact that it includes as special cases the complex algebraic varieties and the Riemann surfaces and furnishes the geometrical basis for functions of several complex variables. Its development has led to clarifications of classical algebraic geometry and to new results and problems. Two notions from algebraic topology have so far played an essential rôle: sheaves (faisceaux) and fiber bundles. But the deeper problems on complex manifolds are not entirely topological.

1. **Topology of complex manifolds.** From the point of view of topology a fundamental problem would be to characterize the orientable manifolds of even dimension $2n$ which can be given a complex structure. But this is too difficult and, at least at the present moment, one

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