## THE NOVEMBER MEETING IN KNOXVILLE

The five hundred nineteenth meeting of the American Mathematical Society was held at the University of Tennessee in Knoxville, Tennessee on Friday and Saturday, November 18–19, 1955. About 130 persons registered, including 82 members of the Society.

By invitation of the Committee to Select Hour Speakers for Southeastern Sectional Meetings, Professor Sze-Tsen Hu of the University of Georgia addressed the Society Friday evening on the subject *Homotopy groups*, with Professor A. D. Wallace presiding.

Sessions for contributed papers were held Friday afternoon and Saturday morning, Professors Wallace Givens, M. K. Fort, Jr., Tomlinson Fort, A. S. Householder, and S. T. Hu presiding.

Abstracts of the papers presented follow. Those having the letter "*t*" after their numbers were read by title. Where a paper has more than one author, that author whose name is followed by "(p)" presented it. Mr. Simon and Mr. Anderson were introduced by Professor A. D. Wallace, Dr. Campbell by Professor J. H. Roberts, and Mr. Kimura by Professor R. J. Koch.

### ALGEBRA AND THEORY OF NUMBERS

75. A. T. Brauer: *On the Schnirelmann density of the sum of two sequences of which one has positive density.*

Let $A$ and $B$ be sequences of positive integers, and $S$ the sum of $A$ and $B$. Assume that the Schnirelmann density $\alpha$ of $A$ is positive and that $B$ is a sequence of order $h$. It was proved by P. Erdös [Acta Arithm. vol. 1 (1936) pp. 197–200] that the Schnirelmann density $\gamma$ of $S$ satisfies the inequality $\gamma \geqq \alpha\{1+(1-\alpha)/2h\}$. E. Landau [*Über einige neuere Fortschritte der additiven Zahlentheorie*, Cambridge, 1937] showed that the order $h$ can be replaced by the mean order $\lambda$. A. Brauer [Math. Zeit. vol. 44 (1938) pp. 212–232] improved this to $\gamma \geqq \alpha\{1+(1-\alpha^{1/2})/\lambda\}$ and S. Selberg [Archiv for Mathematik og Naturvidenskab vol. 47 (1944) pp. 111–118] to $\gamma \geqq \alpha\{1+3(1-\alpha)/4\lambda\}$. Selberg's result is better for $\alpha > 1/9$ and Brauer's result for $\alpha < 1/9$. In this paper these results will be further improved for the case $1/9 \leqq \alpha \leqq 1/2$. (Received October 13, 1955.)

76*t*. Leonard Carlitz: *Resolvents of certain linear groups in a finite field.*

Let $\Gamma$ denote the group of linear transformations $x' = (ax+b)/(cx+d)$ with coefficients in $GF(q)$ and of determinant 1. Dickson has proved that every absolute invariant of $\Gamma$ is a rational function of $J = Q^{(q+1)/2}L^{-(q^2-q)/2}$ $(p>2)$, where $L = x^q - x$, $Q = (x^{q^2}-x)/(x^q-x)$; when $q=2$ the divisor 2 in the exponents is omitted. The equation $J(x) = y$ is normal over $GF[q, x]$ with Galois group $\Gamma$. It is easy to construct a resolvent of degree $q+1$. The principal object of the present paper is the construction of resolvents of lower degree when they occur. It is well known that $\Gamma$ can be represented as a permutation group of degree $\leqq q$ only when $q=5, 7, 9, 11$ in which case the degree is 5, 7, 6, 11, respectively. In each case a resolvent of minimum degree is obtained. In the last part of the paper the ternary linear group is discussed briefly. For $q=2$ the group is of order 168 and a resolvent of degree 8 is constructed; the resolvent of degree 7 in this case is easily obtained. (Received September 23, 1955.)

### 77t. Leonard Carlitz: *The number of solutions of a particular equation in a finite field.*

Faircloth (Canadian Journal of Mathematics vol. 4 (1952) pp. 343–351) has determined explicitly the number of nonzero solutions in $GF(q^2)$ of the equation $x_1^m + \cdots + x_s^m = 1$, where $m \mid q+1$. In the present paper the number of solutions in $GF(q^2)$ of $\alpha_1 x_1^m + \cdots + \alpha_s x_s^m = \alpha$, where $m \mid q+1$ and $\alpha_i$, $\alpha \in GF(q^2)$, is determined. The proof makes use of a result of Stickelberger (Math. Ann. vol. 37 (1890) pp. 321–367). (Received September 23, 1955.)

### 78t. Leonard Carlitz and H. H. Corson: *Some special equations in a finite field.*

Let $N=N(a_1, \cdots, a_r)$ denote the number of solutions of the equation $a_1 x_1^{m_1} + \cdots + a_r x_r^{m_r} = c$ in the finite field $GF(q)$, where $m_i \mid q-1$. Let $A$ denote the average of $N$ for fixed $c$ and fixed $m_i$ and define the variance $\sum \{N(a_1, \cdots, a_r) - A\}^2$ where the sum is over all nonzero $a_i$. When $c=0$ it is shown that $\sum (N-A)^2 = q^{r-2}(q-1)^{r+2}Z$, where $Z$ is a certain function of the $m_i$. A similar result is obtained in the case $c \neq 0$. Necessary and sufficient conditions for the vanishing of $Z$ are determined. More generally similar but more complicated results are obtained for equations of the type $a_1 f_1(u_1) + \cdots + a_r f_r(u_r) = c$, where $u_s = (x_{s1}, \cdots, x_{si_s})$. (Received September 23, 1955.)

### 79. A. H. Clifford and D. D. Miller (p): *Regular $\mathcal{D}$-classes in a semigroup.*

J. A. Green (Ann. of Math. vol. 54 (1951) pp. 163–172) defines the commuting equivalence relations $\mathcal{L}$ and $\mathcal{R}$ in any semigroup $S$ as follows: $a\mathcal{L}b(a\mathcal{R}b)$ if $a$ and $b$ generate the same principal left (right) ideal of $S$. He discusses also their (relative) product $\mathcal{D}=\mathcal{L}\mathcal{R}=\mathcal{R}\mathcal{L}$ and their intersection $\mathcal{L}\cap\mathcal{R}$. Following von Neumann (Proc. Nat. Acad. Sci. U.S.A. vol. 22 (1936) pp. 707–713) the authors say that an element $a$ of $S$ is *regular* if $axa=a$ for some $x \in S$, and they show that either every element or no element of a $\mathcal{D}$-class is regular. Defining an *inverse* of $a$ to be an element $a'$ such that $aa'a=a$ and $a'aa'=a'$, and setting $\mathcal{H}=\mathcal{L}\cap\mathcal{R}$, an $\mathcal{H}$-class $H=L\cap R$ can contain at most one inverse of $a$, and contains exactly one if and only if each of the $\mathcal{H}$-classes $L\cap R_a$ and $L_a\cap R$ contains an idempotent. If $a\mathcal{D}b$, $L_a \cap R_b$ contains an idempotent $e$ if and only if $ab \in L_b \cap R_a$. If $L_b \cap R_a$ also contains an idempotent $f$, and if $H_{11}=\mathcal{H}(e)$, $H_{21}=\mathcal{H}(a)$, $H_{12}=\mathcal{H}(b)$, $H_{22}=\mathcal{H}(f)$, then $H_{ij}H_{jk}=H_{ik}$, and there exist $\mathcal{D}$-classes $D_{12}$ and $D_{21}$ such that $H_{i1}H_{2k} \subseteq D_{12}$, $H_{i2}H_{1k} \subseteq D_{21}$. Other results of a similar nature are obtained, including a theorem which reduces to the Rees-Suschkewitsch Theorem when $S=D$ or $S=D \cup \{0\}$. (Received October 13, 1955.)

### 80t. Eckford Cohen: *The characteristic divisors of a polynomial in a matrix argument.*

Let $A$ represent a matrix in an arbitrary field $F$, and suppose $A$ to have characteristic divisors of the form $Q^h(x)$ where $Q(x)$ is irreducible and separable over $F$. Letting

$f(x)$ represent a polynomial of $F[x]$, the author determines the characteristic divisors of $f(A)$, originally determined in the general case by McCoy (Amer. J. Math. vol. 57 (1935) pp. 491–502). The method employed involves the theory of representation spaces and avoids any use of formal matrix theory. (Received October 6, 1955.)

### 81. P. F. Conrad: *On ordered vector spaces.*

Various ways of ordering a vector space $V$ over an ordered division ring $D$ are considered. Each ordering of $V$ determines an embedding of $V$ into a lexicographically ordered function space. For a given ordering of $V$ let $C$ and $C'$ be convex subspaces such that $C$ covers $C'$. Then $C/C'$ has a natural order, but $C/C'$ is not necessarily $o$-isomorphic to $D$ (where $D$ is considered as a vector space). $V$ is of rank one or $D$ is isomorphic to the field of reals if and only if for all orderings of $V$, every $C/C'$ is $o$-isomorphic to $D$. (Received October 13, 1955.)

### 82. F. A. Ficken: *Basis-free proofs of two theorems on vector spaces.*

It is well known that *every subspace of a vector space has a complement.* This result is established here by applying Zorn's lemma to the class of subspaces not intersecting the given subspace. It is also well known that *the range of the transpose of a linear transformation is the subspace "orthogonal" to the nullspace of the transformation.* Let $L$ map $X$ linearly into $Y$, let $L^*$ be its transpose, mapping $Y^*$ linearly into $X^*$, and let $Y=LX\oplus C$. If $x^*$ is in the subspace of $X^*$ orthogonal to the nullspace of $L$, we define $y^*(Lx+c)$ $(c\in C)$ to be $x^*(x)$ and show easily that $y^*$ is well-defined, $y^*\in Y^*$, and $L^*y^*=x^*$. The rest of the proof is trivial. (Received November 14, 1955.)

### 83. J. R. Garrett: *Reduction of equations to normal form in fields of characteristic $p$.*

In an earlier paper (Duke Math. J. vol. 18 (1951)) the author reduced the equation of degree 5 with coefficients in a field $F$ of characteristic $p=2, 3, 5$ to *principal* and normal form by means of Tschirnhaus transformations. Dickson (*Modern algebraic theories*, Chicago, 1930) gives similar results for the quintic in case $F$ is of characteristic zero which also hold for $F$ of characteristic $p\geqq 7$. This paper extends the previous results to equations of degree $n$, $n\geqq 6$, with coefficients in a field $F$ of characteristic $p\geqq 2$. (Received October 13, 1955.)

### 84. Morikuni Goto: *Lattices of subalgebras of Lie algebras.*

Various results are known about the lattices of subgroups of finite groups. Similar results are obtained in the case of Lie algebras. (Received October 13, 1955.)

### 85. Nickolas Heerema: *Sums of normal endomorphisms.*

Let $N$, $E$, and $A$ represent the sets of normal nilpotent endomorphisms, normal endomorphisms and normal automorphisms, respectively, on a group $G$. Then: (1) $\eta_1-\eta_2\in E$ if $\eta_1\in A$, $\eta_2\in E$; (2) $\eta_1-\eta_2\in E$; $\eta_1+\eta_2\in E$ if $\eta_1\in E$, $\eta_2\in N$. If $G$ has no direct abelian factor and satisfies both chain conditions for normal subgroups then: (3) $\eta_1+\eta_2\in N$, $\eta_1-\eta_2\in N$ if $\eta_1\in N$, $\eta_2\in N$; (4) $\eta_1-\eta_2\in A$ and $\eta_1+\eta_2\in A$ if $\eta_1\in A$, $\eta_2\in N$, and (5) $\eta_1-\eta_2\in N$ if $\eta_1\in A$, $\eta_2\in A$. Proof is based on the fact that a normal endomorphism is idempotent on the commutator subgroup. It is shown that the normal endomorphisms of a group generate a ring $K_G$ of mappings. If $G$ fulfills the above conditions and is indecomposable then $\eta\in K_G$ if and only if $\eta$ is normal and for some positive integer $n$ either $\eta$ or $-\eta$ satisfies (6) and (7) for all $x$ and $y$ in $G$. (6) $\zeta(x)^{-1}y\zeta(x)$

$=x^{-n}yx^n$; (7) $\zeta(xy)=\zeta(x)\zeta(y)(y^{-n}x^{-n}(xy)^n)$. Related results are also obtained. (Received October 6, 1955.)

86*t*. J. H. Hodges: *Distribution of bordered matrices in a finite field.*

Let $q=p^f$ and let $A$, $M$, $U$, $V$ denote matrices with elements in $GF(q)$. If $A$ is nonsingular of order $m$, $U$ is $s\times m$ and $V$ is $m\times t$, put $M=\begin{bmatrix} A & V \\ U & 0\end{bmatrix}$. Then the number of pairs $U$, $V$ such that $M$ has rank $m+r$, $r\leqq s, t$, is determined. The general case where $A$ is an $m\times n$ matrix of rank $h\leqq m, n$ is also considered. The results obtained are analogous to those obtained earlier for $A$ symmetric, skew and hermitian (L. Carlitz and J. H. Hodges, *Distribution of bordered symmetric, skew and hermitian matrices in a finite field*, J. Reine Angew. Math., to appear in 1956). (Received October 13, 1955.)

87. Naoki Kimura: *A decomposition of semigroups containing minimal right ideals.*

Let $S$ be a semigroup containing at least one minimal right ideal, and let $\mathfrak{R}=[R_\gamma:\gamma\in\Gamma]$ be the set of all minimal right ideals. Then $S$ is decomposed into the union of mutually disjoint subsets $C_\phi$ indexed by the transformation semigroup $\Phi$ over $\Gamma$ such that: (1) $C_{\phi\phi'}\subset C_\phi C_{\phi'}$; (2) $\Phi$ contains every primitive projection, i.e., every transformation which sends all points of $\Gamma$ into a single point. Similarly a decomposition of $S$ with at least one minimal right ideal and at least one minimal left ideal is obtained, and the relation between this decomposition and the kernel of $S$ is discussed. (Received October 13, 1955.)

88. G. W. Medlin: *On limits of the real characteristic roots of matrices with real elements.*

In a series of papers published over the past few years A. Brauer has obtained bounds for the characteristic roots of an arbitrary matrix. In this paper it is shown that the bounds for the real roots of matrices with real elements can often be improved. (Received October 6, 1955.)

89. W. M. Perel: *Relative primeness of ordinary integers.*

It is proved that *the probability that a set of r positive integers, chosen arbitrarily, be relatively prime is* $1/\zeta(r)$. ($\zeta$ is the Riemann Zeta-function.) Define $P_r(N)$ to be the number of ordered sets of the form $(X_1, X_2, \cdots, X_r)$ such that (a) the $X_p$ are integers satisfying $1\leqq X_p\leqq N$, where $N$ is a fixed positive integer and (b) the g.c.d. $(X_1, \cdots, X_n)=1$ for each set. The probability that an arbitrary set of $r$ positive integers, all less than or equal $N$, be relatively prime is $P_r(N)/N^r$. The theorem is proved by passing to the limit. (Received October 6, 1955.)

90. Irving Reiner: *Matrix roots of algebraic equations.*

Let $f(x)\in Z[x]$ be a monic $n$th degree polynomial, irreducible over the rational field $Q$, where $Z$ denotes the ring of rational integers. The problem considered here is that of determining the integral classes of $m\times m$ integral matrices $X$ for which $f(X)=0$. Latimer and MacDuffee (Ann. of Math. vol. 34 (1933), pp. 313–316) showed that there are exactly $h$ classes of nonderogatory integral solutions of $f(X)=0$, where $h$ is the class number of $Z[\theta]$, $\theta$ a zero of $f(x)$. Under the hypothesis that $Z[\theta]$ gives all algebraic integers in $Q(\theta)$, it is proved here that the above $h$ classes of solutions of $f(X)=0$ are the only indecomposable classes, so that every integral solution of

$f(X)=0$ is integrally similar to a direct sum of representatives of these classes. The concept of a Maschke representation plays a central role in the proof. (Received September 14, 1955.)

### 91. Chien Wenjen: *On pseudo-normed* algebras.*

A pseudo-normed* algebra $A$ is an algebra over complex numbers $K$ topologized by a family $\mathcal{V}$ of pseudo-norms $V$ such that $V(x)=0$ for $x\in A$ and all $V\in\mathcal{V}$ only if $x=0$ and with an involution * satisfying $V(xx^*)\geqq k_v V(x)V(x^*)$ for all $V\in\mathcal{V}$ $(k_v>0)$. (1) A commutative complete pseudo-normed* algebra is equivalent to a complete subalgebra of the algebra $C(T, K)$ of all continuous complex-valued functions defined on a completely regular, locally compact Hausdorff space $T$ with $k$-topology (compact-open). (2) Let $H=\sum_{V}\in\mathcal{V}\, H_v$ be a Hilbert space defined as the direct sum of the family of Hilbert spaces $H_v$. A complete pseudo-normed* algebra $A$ with $k_v=1$ for all $V\in\mathcal{V}$ can be isomorphically mapped onto a complete subalgebra $A_1$ of the algebra of all linear transformations in a Hilbert space $H=\sum_v\in\mathcal{V}\, H_v$ such that if $x\in A$ maps to $X\in A_1$, then $X$ is bounded in each $H_v$ and $V(x)=\|X\|_v$ for each $V\in\mathcal{V}$, where $\|X\|_v$ denotes the norm of $X$ in the space $H_v$. The spectral theorem for the unbounded self-adjoint linear transformations in a Hilbert space can be proved by means of the first representation theorem. (Received October 6, 1955.)

### 92. R. L. Wilson: *Concerning the matrix equation $AX-XB=C$.*

Solutions of the matrix equation $AX-XB=C$ have been given which require that both $A$ and $B$ be in Jordan form [D. E. Rutherford, Neder. Akad. Wetensch. vol. 35 (1932)] and which require canonical forms for neither of the coefficient matrices [M. H. Ingraham and H. C. Trimble, Amer. J. Math. vol. 63 (1941)]. This paper gives a solution of this equation in which either $A$ or $B$, but not both, are in Jordan form. (Received August 16, 1955.)

## ANALYSIS

### 93. R. C. James: *Linear functionals and reflexivity.*

It is shown that if $B$ has an unconditional basis, then $B$ is reflexive if and only if each linear functional defined on a subspace $B_0$ of $B$ attains its maximum on the unit sphere of $B_0$. Thus $B$ is reflexive if and only if each hyperplane $H$ of $B$ intersects any sphere in $B_0$ which is at zero distance from $H$, where $B_0$ is any subspace of $B$. (Received October 13, 1955.)

### 94. C. W. McArthur: *On relationships amongst certain spaces of sequences in an arbitrary Banach space.*

Let $X$ denote an arbitrary Banach space and $X^*$ denote its adjoint space. A sequence $\{s(i)\}$ in $X$ has an *invariant sum* if and only if there is $x\in X$ such that $x=\lim_{n\to\infty}\sum_{i=1}^{n} s(i)$ and such that $x$ is the sum of each of the convergent rearrangements of $\sum_{i=1}^{\infty} s(i)$. The known result that in any Hilbert space there is a sequence $\{s(i)\}$ with an invariant sum such that $\sum_{i=1}^{\infty} s(i)$ is not unconditionally convergent is extended to any Banach space with infinite dimension. Let $B(X)$ stand for the class of sequences $\{s(i)\}$ in $X$ such that $\sum_{i=1}^{\infty}|f(s(i))|<\infty$ for all $f\in X^*$. Let $B_w(X)$ denote the subclass of $B(X)$ for which there is an $x\in X$ with $f(x)=\sum_{i=1}^{\infty} f(s(i))$ for all $f\in X^*$ and let $B_s(X)$ denote the subclass of $B(X)$ for which there is an $x\in X$ with $x=\sum_{i=1}^{\infty} s(i)$. Let $IS(X)$ stand for the sequences in $X$ with invariant sum and let $U(X)$ denote those sequences $\{s(i)\}$ in $X$ such that $\sum_{i=1}^{\infty} s(i)$ converges unconditionally. For any Banach space $X$, $U(X) \subset B_s(X) = IS(X) \cap B(X) \subset B_w(X) \subset B(X)$, and it is shown that these containments are proper when $X$ is the Banach space, with usual norm, of sequences of real numbers which converge to zero. (Received October 10, 1955.)

### 95*t*. J. S. MacNerney: *Determinants of harmonic matrices.*

This paper is concerned with extensions of a theorem by H. S. Wall [Arch. Math. vol. 5 (1954) pp. 160–167]: if $M$ is a $2 \times 2$ *harmonic matrix* and $F$ corresponds to $M$ then det $M=1$ only in case $F_{11} = -F_{22}$. Theorem 1: If $M$ is an $n \times n$ *harmonic matrix* and $F$ corresponds to $M$ then det $M(s, t) = \exp\left(\sum_1^n [F_{pp}(t) - F_{pp}(s)]\right)$. Theorem 2: If $M$ is an $n \times n$ *quasi-harmonic matrix* and $F$ corresponds to $M$ and $G$ is the continuous part of $F$ then det $M(s, t) = \exp\left(\sum_1^n [G_{pp}(t) - G_{pp}(s)]\right)$. For the notion of a *quasi-harmonic matrix*, see Bull. Amer. Math. Soc. Abstract 61-1-99 [or paper, to appear in J. Elisha Mitchell Sci. Soc. vol. 71 (1955)]. (Received October 13, 1955.)

### 96*t*. J. A. Nohel and W. K. Ergen: *Real representations of solutions of a linear functional-differential equation.*

Consider the equation (1) $y'(t) = -\omega^2/\theta \int_0^\theta (\theta - h) y(t-h) dh + W(t)$, $(' = d/dt)$, $t \geqq \theta > 0$, where $y$, $t$, and $W$ are real, and $\omega^2$ and $\theta$ are real constants. Let $y(t)$ satisfy the given initial condition (2) $y(t) = q(t)$, $0 \leqq t \leqq \theta$. The following result is established: Let $q(t)$ be continuous on $0 \leqq t \leqq \theta$. Let there exist constants $c_1$ and $c_2$ such that $|W(t)| \leqq c_1 \exp(c_2 t)$, $t \geqq \theta$; then the solution of (1) satisfying (2) is given by (3) $y(t) = q(\theta) \cdot K(t-\theta) - \omega^2/\theta \int_0^\theta \{\int_0^r (r-\rho) K(t-\theta-\rho) d\rho\} q(r) dr + \int_\theta^t K(t-\tau) W(\tau) d\tau$, $t > \theta$, where (4) $K(t) = \lim_{T \to +\infty} (1/2\pi i) \int_{b-iT}^{b+iT} \exp(ts) 1/\{s + \omega^2/s - (\omega^2/\theta s^2)[1 - \exp(-\theta s)]\} ds$, $b$ sufficiently large. Moreover, if $\omega \neq 2n\pi/\theta$, $n = \pm 1, \pm 2, \cdots$, there exists a constant $\lambda > 0$, such that $K(t) = O(\exp(-(\lambda/2)t))$ as $t \to +\infty$. This result is needed in the study of stability of solutions of nonlinear equations similar to (1) where $W(t)$ is replaced by a nonlinear "delay" term; see Nohel and Ergen: *Stability of solutions of a nonlinear functional differential equation*, Bull. Amer. Math. Soc. Abstract 62-1-97. Laplace transforms are used throughout and the results are similar to those of R. Bellman for differential-difference equations. (Received October 3, 1955.)

### 97. J. A. Nohel (p) and W. K. Ergen: *Stability of solutions of a nonlinear functional-differential equation.*

Consider the equation (1) $x'(t) = -\omega^2/\theta \int_0^\theta (\theta-h) x(t-h) dh - c/\theta \int_0^\theta (\theta - h) g[x(t-h)] dh$ $(' = d/dt)$, $t \geqq \theta > 0$, where $t$, $x$, $g$ are real, and $\omega^2$, $c$, $\theta$ are positive constants, and $x(t)$ satisfies the given initial condition (2) $x(t) = q(t)$, $0 \leqq t \leqq \theta$. The following result is established: Let $q(t)$ be continuous on $0 \leqq t \leqq \theta$; let $g(x)$ be continuous for all $x$ with $g(x) = o(|x|)$ $(|x| \to 0)$, uniformly in $t$, $t \geqq \theta$; then if $\omega \neq 2n\pi/\theta$, $n = \pm 1, \pm 2, \cdots$, the identically zero solution of (1) is asymptotically stable, hence stable. Note that the concept of stability first has to be defined. In the proof a real representation theorem for linear equations of the type (1), see Nohel and Ergen: *Real representations of solutions of a linear functional-differential equation*, Bull. Amer. Math. Soc. Abstract 62-1-96, is used to convert (1) to a workable integral equation. The situation here is similar to the case of differential-difference equations studied extensively by R. Bellman. A somewhat sharper result requiring the more stringent hypothesis $|g(x_2) - g(x_1)| \leqq \epsilon |x_2 - x_1|$, for $|x_1|$ and $|x_2|$ sufficiently small and where $\epsilon \to 0$ as $|x_1|$ and $|x_2|$ tend to 0, is also given. (Received October 3, 1955.)

### 98. B. J. Pettis: *On Moore-Smith limits and filtered functions.*

The purpose of the present paper is to present for set nets in product spaces theorems that yield in topological algebras the usual "limit of the sum (product) is the sum (product) of the limits" theorems for integrands, functions, derivatives, etc. (Received October 13, 1955.)

### 99t. Marvin Rosenblum: *Perturbation of the continuous spectrum and unitary equivalence.*

Let $H$ be a Hilbert space and let $A$ and $B$ be possibly unbounded self-adjoint operators in $H$ such that $B-A=\sum_{j=1}^{\infty}\lambda_j(\cdot,\phi_j)\phi_j$, where the $\phi_j$ are orthonormal and $\sum_{j=1}^{\infty}|\lambda_j|<\infty$. Suppose that the spectral measure of $A$ is weakly absolutely continuous. Theorem: $B$ is unitarily equivalent to $A$ if and only if the spectral measure of $B$ is weakly absolutely continuous. This generalizes the result quoted in Bull. Amer. Math. Soc. Abstract 61-1-155. (Received October 6, 1955.)

### 100. F. B. Wright, Jr.: *Note on a theorem of Hille and Zorn.*

Hille and Zorn (Ann. of Math. vol. 44 (1943) pp. 554–561) characterized the open additive semigroups of the plane $E^2$ by means of functions $f$ which satisfy the functional relation $f(x+y)\leqq f(x)+f(y)$. Rosenbaum (Duke Math. J. vol. 17 (1950) pp. 227–247) extended this result to Euclidean $n$-space $E^n$. Their characterization depends on a suitable but not unique choice of basis in the vector space. In this note a characterization is given which does not depend on a choice of basis and which furthermore extends to any real linear topological space $E$. Call an extended real-valued function $f$ on $E$ "submodular" if $f(x+y)\leqq\max(f(x),f(y))$. Then there is a one-one correspondence between open semigroups $S$ in $E$ and upper semicontinuous submodular functions $f$ on $E$, provided $f$ is suitably normalized. This correspondence is given by the relation $S=[x\in E:0\leqq f(x)<1]$. A necessary and sufficient condition is given that $f$ define an angular semigroup $S$. (Received October 11, 1955.)

## Applied Mathematics

### 101. S. G. Campbell: *Numerical evaluation of certain statistical integrals.*

For numerical purposes it may be desirable to evaluate the function $w=f(z)$ for $z$ in some domain $D$ of a metric space by evaluating the approximating function $w^*=F[g(z^*)]$. Multiple decision procedure determines the proper transformation $z^*=z^*(z)$ which allows $g(z^*)$ to be calculated with optimum speed and accuracy, and also determines the mapping $F$, thus trading slow arithmetic operations for fast decision procedure. Comparison of such programs is effected by defining the relative computing efficiency $H(f)$ in terms of certain multiple integrals whose integrands turn out, even in simple cases, to be implicit solutions of transcendental equations. Optimal bounds $N_1(h)$ and $N_2(h)$ are constructed such that $N_1(h)\leqq H(f)\leqq N_2(h)$ where the integrand $h$ is a step function related to $f$; these bounds are difficult to calculate, particularly in programs with variable multiple induction loops, and do not give the desired statistical information. Accordingly, a scheme of statistical integration is defined in which the computing program itself becomes the integrand and the quantities $N_1(h)$, $N(f)$, $N_2(h)$ may be calculated as accurately as desired. This procedure has been applied effectively to several computing programs; examples are given. (Received October 13, 1955.)

102. W. S. Loud: *Nonlinear systems with limiting and large-amplitude forcing.*

Nonlinear systems of the form $\ddot{x}+2b\dot{x}+(a^2+b^2)x=\phi(A\cos t-\alpha x-\beta\dot{x})$ where $\alpha$, $\beta$, $a^2$, $b$ and $A$ are real, $\phi(u)=1$ if $u\leqq -1$, $u$ if $|u|\leqq 1$, and $-1$ if $u\geqq 1$, are studied for large values of the amplitude $A$. It is shown that with minor exceptions that for large $A$ there exists a unique solution of the system, $x_0(t)$, with period $2\pi$. Moreover given an integer $n$ there exists a number $A_0$ depending on $n$ such that for $A>A_0$ there is no solution of period $2n\pi$ other than $x_0(t)$, so that proper subharmonics do not exist. The exceptional case is the case $b=0$ with $na$ an integer. Use is made of the substitution $x=y/k$ $(k=1/A)$ and the degenerate system $\ddot{y}+2r\dot{y}+(a^2+b^2)y=-k\cdot\operatorname{sgn}(\cos t-\alpha y-\beta\dot{y})$. This work was supported in part by the Office of Ordnance Research. (Received October 6, 1955.)

103. E. P. Miles, Jr. (p) and Ernest Williams: *The Cauchy problem for linear partial differential equations with restricted boundary conditions.* I.

Consider a linear partial differential equation $^*\Phi(D, x_1, x_2, \cdots, x_n)u+\Psi(D,t)u=0$, $\Psi$ an ordinary differential operator of order $s$ with respect to $t$. A solution of $*$ for the Cauchy data $u(x_1, x_2, \cdots, x_n, 0)=P(x_1, x_2, \cdots, x_n)$, $\partial^j u(x_1, x_2, \cdots, x_n, 0)/\partial t^j=0$, $j=1, 2, \cdots, s-1$, where $P$ is annihilated by a $(k+1)$-fold iteration of the operator $\Phi$ is shown to be $\sum_{j=0}^{k}\Phi^j(P)\cdot u_j$; where the $u_j$ are a set of solutions of the system of ordinary differential equations $^{\#}\Psi(u_j)+u_{j-1}=0$, $j=1, 2, \cdots, k$, $\Psi(u_0)=0$ with initial conditions $u_0(0)=1$, $u_j(0)=0$, $j\geqq 1 d^m u_j(0)/dt^m=0$ for $m=1, 2, \cdots, s-1$ and all $j$. An analogous solution for $*$ and the Cauchy Data $u(0)=0$, $u_t(0)=Q$, $s=2$ is obtained. These methods lead to the following improved form of the authors' basic set of solutions of the wave equation (Proc. Amer. Math. Soc. vol. 6, pp. 191–194). For each set of non-negative integers $a$, $b$, $c$, $d$, $a+b+c+d=n$, $d\leqq 1$; $W_{a,b,c,d}=\sum_{j=0}^{[n/2]}\nabla^{2j}(x^a y^b z^c)\cdot t^{2j+d}/(2j+d)$. (Received October 11, 1955.)

104. E. P. Miles, Jr. and Ernest Williams (p): *The Cauchy problem for linear partial differential equations with restricted boundary conditions.* II.

The method of the preceding abstract applied to the Euler-Poisson-Darboux equation $^*\nabla^2 u-[u_{tt}+kt^{-1}u_t]=0$ with $u(x, y, z, 0)=P(x, y, z)$ gives, for $k>0$, a basic set of solutions similar to that given for the wave equation with $d=0$ and $t^{2j}/(2j)!$ replaced by $t^{2j}/(1+k)(3+k)\cdots(2j-1+k)\cdot 2^j j!$. If $k<0$ the system $^{\#}$ associated with the solution of $*$ has a solution $u_j=\sum_{n=0}^{j}a_n t^{2j-2n+1-k}/2^{j-n}(j-n)!(1-k)\cdots(2j-2n+1-k)+t^{2j}/2^j j!(1+k)(3+k)\cdots(2j-1+k)$, $k\neq -1, -3, \cdots, -(2j-1)$, which is not unique since the $a_n$ are arbitrary. These results for the E.P.D. equation are in agreement with those of A. Weinstein (Proceedings of the Fifth Symposium in Applied Mathematics, McGraw-Hill, 1954). For the vibrating string problem $a^2u_{xx}-u_{tt}=0$, $u(x, 0)=P(x)=\sum_{n=1}^{\infty}b_n\sin(n\pi x/L)$; $u_t(x, 0)=0$; $u(0, t)=u(L, t)=0$ the method leads to a formal solution which, after interchanging the order of summation, takes the usual form $u=\sum_{n=1}^{\infty}b_n\sin(n\pi x/L)\cos(n\pi at/L)$. (Received October 11, 1955.)

105. N. C. Perry and J. C. Morelock (p): *A statistical approach to the propagation of round-off error by division.*

In this paper a detailed derivation is given of the error in a quotient $a/b$ caused by relatively large errors $\epsilon$ and $\epsilon'$ respectively in the numerator and denominator. $a$ and $b$ are constant and $\epsilon$ and $\epsilon'$ are assumed to be independent errors distributed in a rectangular distribution on the range from $-1/2$ to $+1/2$. This result is related to the work of Inman [Math. Gazette (1950)], who, however, takes $a$ and $b$ to be random variables which are large relative to $\epsilon$ and $\epsilon'$, and tends to give results only. (Received October 11, 1955.)

### 106t. J. P. Roth: *An application of algebraic topology to numerical analysis.* III. *Three theorems on networks.*

1. Let $K$ be a 1-complex, $L$ an isomorphism of the space of 1-chains of $K$ (complex coefficient field) on the dual space of 1-cochains; $L$ is said to be *ohmic* if $v\neq 0$ implies $vLv\neq 0$: the proof (Proc. Nat. Acad. Sci. U.S.A. vol. 41 (1955) pp. 518–521) for the existence of a solution to the "electrical" network problem actually utilized only the condition that $L$ be ohmic, not the stronger, that $L$ be power definite. 2. Proposition. Any linear system of equations can be represented as a network problem. 3. Proposition. Power is invariant under interconnection. The problem of this invariance has plagued the literature for 25 years. It is shown here to be a simple consequence of the transformations induced by the simplicial map which is the interconnection rather than the reverse as was previously attempted. (Received October 13, 1955.)

### 107. W. C. Royster (p) and S. D. Conte: *Finite difference approximations to a fourth order parabolic differential equation.*

Consider the fourth order parabolic differential equation (*) $u_x^4+u_{tt}=0$, $0<x<1$, $t>0$, subject to the conditions $u(0, t)=u(1, t)=u_{xx}(0, t)=u_{xx}(1, t)=0$, $t>0$, $u(x, 0)=f(x)$, $u_t(x, 0)=g(x)$. The exact solution of (*) is $u(x, t)=\sum_{n=1}^{\infty} a_n \sin n\pi x \cos n^2\pi^2 t$, where $an=2\int_0^l f(x)\sin n\pi x dx$ and $f(x)$ is assumed to satisfy certain continuity conditions. The forward difference approximation of (*) is (**) $\Delta_t^2 u(x, t)=-r^2\Delta_x^4 u(x, t)$ where $\Delta_x^4$ and $\Delta_t^2$ denote the fourth central difference with respect to $x$ and the second central difference with respect to $t$, respectively, and $r$ is the mesh ratio defined by $\Delta t=r(\Delta x)^2$. Collatz (ZAMM vol. 31 (1951)) has shown that (**) is stable when $r\leqq 1/2$. It is shown in this paper that the solution of (**) converges for all $r>0$ to the solution of (*) for certain restrictions on $f(x)$. An implicit difference approximation (***)$\Delta_t^2 u(x, t)=-r^2/4\Delta_x^4[u(x, t+\Delta t)+2u(x, t)+u(x, t-\Delta t)$ is shown to be stable for all $r$. The solution of (***) converges to the solution of (*) under the same conditions on $f(x)$ as for the forward difference scheme. The numerical solution of (***) requires the solution of a system of $m-1$ equations in $m-1$ unknowns, where $m\Delta x=1$, which must be solved at each $t$ line in the grid. A procedure which employs a relatively small amount of machine time is given whereby the values of $u$ can be computed. (Received October 13, 1955.)

### 108. W. S. Snyder: *A method of obtaining bounds for the solution of Volterra integral equations.*

If the kernel of a Volterra integral equation is non-negative and if values of a function $f$ are not increased at any point of the range by iteration in the equation, then $f$ and its iterate are upper bounds for the solution. A similar statement holds for lower bounds of the solution. A procedure is outlined by which such upper and lower bounds can be generated by a digital computer. These bounds may diverge rapidly in certain cases but tend to a common limit as the mesh size used for the integration tends to zero if the kernel is sufficiently smooth. The method seems practical for a wide class of Volterra equations, differential-difference equations, and differential equations. The advantage of the method is that the computing machine in effect does the error analysis, and the round-off errors may be handled to give completely rigorous bounds for the solution. (Received October 20, 1955.)

## TOPOLOGY

### 109. L. W. Anderson: *A theorem on topological chains.*

A topological lattice is a pair of continuous functions $\bigvee: L\times L\rightarrow L$ and $\bigwedge: L\times L\rightarrow L$ where $L$ is a Hausdorff space and $\bigvee$ and $\bigwedge$ satisfy the usual conditions stipulated for a lattice. It is shown that a compact connected one-dimensional topological lattice is a chain. Cohomological methods are required to prove this result. It is also shown that a compact connected topological chain is one-dimensional. (Received October 13, 1955.)

### 110. T. R. Brahana: *Products of quasi-complexes.*

It is shown that the product space $X\times Y$ of two quasi-complexes $X$ and $Y$ is a quasi-complex. Quasi-complexes are spaces defined by Lefschetz [*Algebraic topology*, p. 322], in connection with the theory of fixed points, and as a consequence of this result, the Lefschetz fixed point theorem holds for products of quasi-complexes. The proof consists in showing the constructability of projections and anti-projections among the nerves of product coverings of $X\times Y$. Projections can be constructed canonically. Given two anti-projections $\omega_{\beta\alpha}$ and $\bar{\omega}_{\beta'\alpha'}$ on nerves of coverings of the spaces $X$ and $Y$, anti-projections are constructed by forming a chain mapping of the chains of the nerve of the product covering onto the chains of the cartesian product [Eilenberg-Steenrod, *Foundations of algebraic topology*, p. 66], then mapping these chains by means of a suitable chain mapping to the subgroup generated by chains of the form $\sigma^p\times\tau^q$ (cartesian product of simplexes), and then applying the map $(\omega_{\beta\alpha}\times\bar{\omega}_{\beta'\alpha'})$ to this subgroup. The composition map is the desired anti-projection. (Received October 13, 1955.)

### 111. C. E. Capel (p) and W. L. Strother: *A new proof of a fixed-point theorem of Wallace.*

A. D. Wallace [Bull. Amer. Math. Soc. vol. 47 (1941) pp. 757–760] has shown that if $T$ is a tree (in a combinatorial sense) and $F$ is a continuous, multi-valued function on $T$ to itself, such that the image of each point is connected, then $F$ has a fixed point. Using L. E. Ward's characterization of a tree in terms of a partially ordered space [Proc. Amer. Math. Soc. vol. 5 (1954) pp. 992–994], an order-theoretic version of Wallace's theorem is given. (Received October 10, 1955.)

### 112. C. E. Capel and W. L. Strother (p): *Knaster continua.*

Let $X$ denote a compact connected metric space. A continuous multi-valued point connected function $F$ on the unit interval to a space $X$ will be called an $M$-arc in $X$. Utilizing Kelley's results on hyperspaces [*Hyperspaces of a continuum*, Trans. Amer. Math. Soc. vol. 52 (1942)] indecomposability can be characterized as follows: A compact metric space $X$ is indecomposable if and only if there is a pair of connected closed sets $A$ and $B$ in $X$ such that for every $M$-arc $F: I\rightarrow X$ joining $A$ and $B$ there is a point $a$ in $I$ such that $F(a)$ is the whole space $X$. Also $X$ is a Knaster continuum if and only if $M$-arcs in $X$ are unique. (Received October 10, 1955.)

## 113. Haskell Cohen: *Fixed points in products of ordered spaces.*

Let $X$ and $Y$ be compact connected ordered topological spaces and $Z=X\times Y$. It is shown that $Z$ is unicoherent and that if $A$ is a closed set in $Z$, either some component of $A$ projects onto $X$ or a component of the complement of $A$ projects onto $Y$. Using these properties it is shown that $Z$ has f.p.p. (the fixed point property; i.e. every continuous function from $Z$ to itself has a fixed point). This last result can be rephrased and strengthened to read: The product of two ordered spaces with f.p.p. has f.p.p. That this is actually a strengthening can be seen by means of an unpublished example due to C. T. Yang of a noncompact ordered space with the fixed point property. (Received October 11, 1955.)

## 114. H. C. Griffith: *Tamely imbedded curves in three-space.*

A 1-manifold $J$ in Euclidean 3-space $R$ has property $\mathcal{P}_t$ at the point $x\in J$ provided in each neighborhood of $x$ there is a topological 2-sphere $K$ enclosing $x$ which is locally tame modulo $J$ and meets $J$ in a set of cardinality equal to the order of $x$ in $J$. Also, $J$ has property $\mathcal{Q}_t$ at $x$ provided there is a topological disk $D$ which is locally tame modulo $J$ and meets $J$ in the closure of a neighborhood (relative to $J$) of $x$. It is shown that having property $\mathcal{P}_t$ (property $\mathcal{Q}_t$) at $x$ is equivalent to having property $\mathcal{P}$ (property $\mathcal{Q}$) at $x$, where $J$ has property $\mathcal{P}$ (property $\mathcal{Q}$) at $x$ provided the 2-sphere $K$ (the disk $D$) exists as above and in addition is locally polyhedral modulo $J$. Thus results involving properties $\mathcal{P}$ and $\mathcal{Q}$ remain valid when properties $\mathcal{P}_t$ and $\mathcal{Q}_t$ are substituted. For example, a necessary and sufficient condition that an arc or simple closed curve be tamely imbedded in $R$ is that it have both property $\mathcal{P}_t$ and property $\mathcal{Q}_t$ at each of its points (see Harrold, Griffith, and Posey, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954)). (Received October 7, 1955.)

## 115. O. G. Harrold, Jr.: *Some consequences of the recent approximation theorem of Bing.*

By the use of an approximation theorem proved by R. H. Bing (*Approximating wild surfaces with polyhedral ones*) and results of *A characterization of tame curves*, Trans. Amer. Math. Soc. vol. 79 (1955) pp. 12–34, the following results are established. An arc in 3-space is contained in the interior of a 2-cell if and only if it lies on the boundary of some bounded, open, connected, ulc[1] subset of space. If a simple closed curve is polyhedral and bounds a disk in space, it bounds a polyhedral disk. Finally, the requirements that certain sets referred to in the definitions of properties $\mathcal{P}$ and $\mathcal{Q}$ discussed in the paper mentioned above be "almost polyhedral" may be dropped completely. (Received October 10, 1955.)

## 116. J. G. Horne, Jr.; *On semi-groups and compactification theorems.* Preliminary report.

P. S. Alexandrov defined a topology for the set $\gamma(X)$ of completely regular ends in the family of open subsets of a completely regular space $X$, and showed $\gamma(X)$ homeomorphic to the Stone-Čech compactification $\beta(X)$of $X$ (Rec. Math. [Mat. Sbornik] N.S. vol. 5 (47) (1939) pp. 403–423). For an arbitrary space $X$ we show directly that $\gamma(X)$ is a compact Hausdorff space and that every continuous real function on $X$ has unique extension to $\gamma(X)$. For this, we use the notion of *p-ideal.* A *p-ideal* in a commutative semi-group is an *o-ideal* $M$ of Milgram (Duke Math. J. vol. 16, p. 377) having the additional property: if $e$ is a unit for $f\notin M$ then $ek\in M$ only if $k\in M$. Sufficient conditions are obtained in order that the space of $p$-ideals be com pact. Equivalence classes of $p$-ideals are defined and a one-to-one correspondence exhibited between equivalence classes of $p$-ideals in the bounded continuous functions under multiplication (lattice "cap") and $\gamma(X)$. It follows that if $X$ is completely regular then the bounded continuous on $X$ under multiplication (lattice "cap") characterize $\beta(X)$. This generalizes a theorem of Milgram (this is Blair's generalization of Kaplansky's result (Bull. Amer. Math. Soc. Abstract 61-3-436)). (Received October 13, 1955.)

### 117. R. J. Koch (p) and A. D. Wallace: *Notes on a paper of J. A. Green.*

A mob $S$ is a Hausdorff topological semigroup. We say that $S$ is stable if (i) $a, b \in S$ and $Sa \subset Sab$ imply $Sa = Sab$ and (ii) $a, b \in S$ and $aS \subset baS$ imply $aS = baS$. Using stability in place of minimum conditions on principal left and right ideals, J. A. Green's Theorem 8 [Ann. of Math. vol. 54 (1951)] is proved, so that in a stable mob each simple element is regular, and $d$-classes coincide with $f$-classes. If $S$ has a unit and satisfies the above mentioned minimum conditions then $S$ is shown to be stable. Further, the class of stable mobs is shown to include compact mobs, commutative mobs, mobs which are the union of groups, and mobs in which each orbit closure $(x, x^2, x^3, \cdots)^*$ is compact and each principal ideal has an idempotent generator. (Received October 13, 1955.)

### 118. Ceslovas Masaitis: *Complement of a finite dendrite in the compactified 3-space.* Preliminary report.

The following result is proved: If a dendrite $d$ can be represented as a finite union of simple arcs in a compactified 3-space $R$ and if $d$ has property P then $R \backslash d$ is an open 3-cell. (The property P is defined in *The enclosing of simple arcs and curves by polyhedra* by O. G. Harrold, Jr., Duke Math. J. vol. 21, pp. 615–622. The technique of the proof of the result above is essentially that of the paper quoted.) Let $\{B_\alpha\}$, $\alpha = 1, 2, \cdots, \alpha'$, be the set of branchpoints of $d$ and $r(B_\alpha)$ be the order (Menger-Urysohn) of $B_\alpha$ in $d$. The set of disjoint 2-spheres $K(B_\alpha)$ each meeting $d$ in $r(B_\alpha)$ points is constructed. If int $K(B_\alpha)$ denotes the bounded component of $R \backslash K(B_\alpha)$ (the complement of $K(B_\alpha)$ in $R$) then $a_\alpha = d \cap \text{int } K(B_\alpha)$ and $\{d_{\alpha\beta}\}$, $\beta = 1, 2, \cdots, r(B_\alpha)$, is the set of components of $d \backslash \bigcup_{\alpha=1}^{\alpha'} a_\alpha$ with the end-points on $K(B_\alpha)$. Further, $S(d, \epsilon) = \{A \mid \rho(A, d) < \epsilon\}$. $K_{\tau-1,\sigma-1}$ is a 2-sphere which has the following properties: (i) $K_{\tau-1,\sigma-1} \subset S(d_1 \epsilon)$, (ii) $\bigcup_{\alpha=1}^{\tau-1} (a_\alpha \cup \bigcup_{\beta=1}^{r(B\alpha)} d_{\alpha\beta}) \cup a \cup \bigcup_{\beta=1}^{\sigma-1} d_\beta \subset K_{\tau-1,\sigma-1}$, (iii) $K_{\tau-1,\sigma-1}$ is locally polyhedral mod $d$, (iv) $K_{\tau-1,\sigma-1} \cap d$ consists of finite number of points. It is shown by induction on $\tau$ and $\sigma$: (a) that $K_{\tau-1,\sigma-1}$ exists for every $\tau \leqq \alpha'$ and $\sigma \leqq r(B_\tau) + 1$, (b) that if $K = K_{\alpha'-1, r(B\alpha')}$ then $d \subset \text{int } K$. Hence $d \cap K = \square$ and therefore $K$ is polyhedral. Repetition of the argument of Theorem 3 in the paper quoted above proves the assertion. (Received October 10, 1955.)

### 119. P. S. Mostert (p) and A. L. Shields: *On a class of semigroups on $E_n$.*

Theorem A. Let $S$ be the half-line $[0, \infty)$. Suppose $S$ is a (topological) semigroup with zero at 0 and identity at 1. Then (i) if $S$ contains no other idempotents, its multiplication is the ordinary multiplication of real numbers on $[0, \infty)$; (ii) if $S$ contains an idempotent different from 0 and 1, then it contains a largest (in the sense of the regular order of real numbers) such idempotent $e$. Moreover, $e < 1$, $[e, \infty)$ is a subsemigroup iseomorphic to $[0, \infty)$ under the usual multiplication of real numbers, and $[0, e]$ is an $(I)$-semigroup. (The complete structure of $(I)$-semigroups has been obtained by the authors.) (Iseomorphism = simultaneous isomorphism and homeo morphism.) Theorem B. Let $S$ be a semigroup with identity on $E_n$, $n>1$, and $B$ a compact connected submanifold of dimension $n-1$. If $B$ is a subsemigroup containing the identity of $S$, then (i) $n=2$ or 4 and $B$ is a Lie group which is $S^1$ if $n=2$ and $S^3$ if $n=4$ (where $S^i$ denotes the $i$-sphere); (ii) there exists a subsemigroup $J$ contained in the center of $S$ which is iseomorphic to a semigroup of the type described in Theorem A; (iii) the subsemigroup $J$ meets each orbit $xB=bx$ of $B$ in exactly one point, and $JB=S$; (iv) if 0 denotes the zero for $J$, then 0 is a zero for $S$, and $(J-\{0\})\times B$ is iseomorphic to $(J-\{0\})B=S-\{0\}$ in the natural way. (Received October 13, 1955.)

120. P. S. Mostert and A. L. Shields (p): *On continuous multiplications on the two-sphere.*

By a semigroup we mean a topological semigroup, that is, a Hausdorff space with a continuous, associative multiplication. A. D. Wallace has shown that a compact connected manifold which is a semigroup with identity must be a group. Hence the two-sphere cannot be a semigroup with identity. S. T. Hu has raised the question whether the two-sphere can be a semigroup with a circle subgroup. The answer is yes, and one can give both abelian and non-abelian examples. The following theorem can be proven. Theorem: Let $S$ be a semigroup which is topologically the two-sphere, and suppose $B$ is a circle subgroup. Let $S\backslash B=A\cup C$ (the symbol $\backslash$ denotes set-theoretic difference). Then, (i) $S$ has a zero, 0, and $0\notin B$. (ii) If $0\in C$, then $Q=C\cup B$ is an ideal of $S$, and the elements of $Q$ commute with every element of $S$. (iii) There exists an element $a\in A$, $a\notin SS$, such that $ax=xa=0$ for every $x\in S$. Remark: $Q$ is a two-cell semigroup in which the boundary is a group. The authors have previously classified all the possible multiplications for $Q$. (Received October 13, 1955.)

121. A. D. Wallace: *The structure of one-dimensional Peano clans.*

A mob is a Hausdorff space together with a continuous associative multiplication. A clan is a compact connected mob with unit. A Peano space is one which is compact, connected, locally connected and metrizable. It is known that a one-dimensional homogeneous clan is a group. In this note it is shown inter alia that a one-dimensional Peano clan is either a tree ( = dendron, dendrite, acyclic continuous curve) or contains just one simple closed curve which is the minimal ideal of the clan. It is shown that the well-known dyadic tree will support a continuous associative multiplication so that its set of end points is also the maximal subgroup containing the unit. (Received October 5, 1955.)

122. A. B. Simon: *n-cyclic elements.* Preliminary report.

In this note some of the results of G. T. Whyburn on *Cyclic elements of higher order* are generalized to the case of arbitrary compact Hausdorff spaces and any non-zero coefficient group. The main result obtained is the following: Let $(X, X_0)$ be a compact Hausdorff pair and fix a coefficient group, $G$. Let $\{K_\delta^{n-1}\}_{\delta\in\Delta}$ be the collection of all $n-1$ cyclic elements (mod $X_0$) of $X$. There exists a closed subset, $A$, of $X$ such that if $U$ is open about $A$ then, for all but a finite number of $\delta\in\Delta$, $K_\delta^{n-1}\subset U$. If this set $A$ is a $T^{n-1}$ (mod $X_0$), i.e. the $n-1$ groups of $A$ are trivial, then $I^*: H^n(X, X_0) \approx \sum_{\delta\in\Delta} H^n(K_\delta^{n-1}, K_\delta^{n-1}\cap X_0)$ where $I^*$ is the product of the $i_\delta^*$ induced by inclusion and $\sum_{\delta\in\Delta}$ is the weak cartesian product. (Received October 13, 1955.)

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